Happening Today

• Calendar updated on Stellar
• Trouble shoot session after class
• Project 1 will be posted
• Online project 20:00:00
Finite difference in 1D

• Error and stability
• How to build a scheme

Finite difference in 2D & 3D

• 2D Poisson’s equation
• Matrix form
• 3D Poisson’s equation
• Error and stability
Truncation error and solution error

\[ e := \hat{u} - u \]
\[ A\hat{u} = b \quad \text{discretized PDE} \]
\[ A e = \tau \quad \text{error equation} \]
\[ \tau := Au - \frac{\delta^2 u}{\delta x^2} \]
\[ ||e|| \leq ||A|| ||u|| \]
How to construct FD operators?

\[ \frac{\partial u}{\partial x} \bigg|_{i} = W_{-2} u_{i-2} + W_{1} u_{i-1} + W_{0} u_{i} + W_{1} u_{i+1} \]

**Apply Taylor**

\[ W_{-2} u_{i-2} = w_{-2} u_{i} - (20x) \frac{\partial u}{\partial x} \bigg|_{2} + \frac{w_{2}(20x)^{2}}{2} \frac{\partial^{2} u}{\partial x^{2}} - \frac{(20x)^{3}}{6} \frac{\partial^{3} u}{\partial x^{3}} + O(\Delta x^{4}) \]

\[ W_{-1} u_{i-1} = w_{-1} u_{i} + \Delta x \]

\[ W_{0} u_{i} = w_{0} \]

\[ W_{1} u_{i+1} = w_{1} u_{i} + \Delta x \]

\[ \frac{\partial^{2} u}{\partial x^{2}} = \frac{w_{2}}{2} \]

\[ \frac{\partial^{3} u}{\partial x^{3}} = \frac{w_{3}}{6} \]

\[ \frac{\partial^{4} u}{\partial x^{4}} = \frac{w_{4}}{24} \]
\[ \begin{pmatrix} w_2 & w_1 & w_0 & v_i \end{pmatrix} \begin{pmatrix} 6x & \frac{\partial x^2}{2} & \frac{\partial x^3}{6} \\ 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = (0, 1, 0, 0) \]

\[ w_2 = \frac{1}{6\Delta x} \quad w_1 = -\frac{1}{6x} \quad w_0 = \frac{1}{2\Delta x} \quad v_i = \frac{1}{3\Delta x} \]
Finite Difference for Multi-D Elliptic Partial Differential Equations
Finite Difference for Multi-D Elliptic Partial Differential Equations

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y) = 0 \]

\[ u(x, y) = 0 \quad \text{at} \quad \Gamma \]
FD Approximation of 2D Poisson Equation

1. \( \hat{f}_{i,j} = f(x_i, y_j) \)

2. Discretize PDE into

   \[
   \sum_{(i,j)} A_{i,j} \hat{u}_i + \hat{f} = 0
   \]

   \[
   \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
   \]

3. Use \( \hat{u} \) to approximate \( u \)
Matrix form

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \begin{cases} 
\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}
\end{cases}
\]

when \( \Delta x = \Delta y \)

\[
\frac{\partial^2 u}{\partial x^2} = \begin{cases} 
\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}
\end{cases}
\]

\[
\frac{\partial^2 u}{\partial x^2} \leq \frac{u_{i+1} - 2u_{i} + u_{i-1}}{\Delta x^2}
\]
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\Delta x^2} \sum_{i,j} U_{i,j} \]