Error equation for NONLINEAR, UNSTEADY PDE

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \]

CFL \# \frac{\Delta t}{\Delta x}

Ratio of eigenvalues of spatial discretization
To stability region of explicit time integrator

\[ \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} = 0 \]
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0
\]

Linearization:
\[
f(u) - f(\hat{u}) = \frac{df}{du}(u - \hat{u})
\]

\[
\frac{\partial u_i}{\partial t} + \hat{u}_i \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0
\]

\[
\frac{\partial u_i}{\partial t} + u_i \frac{u_{i+1} - u_{i-1}}{2\Delta x} = \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right)_{i}
\]

\[
e_i := u_i - \hat{u}_i
\]

\[
\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(\hat{u} \cdot e) = \gamma
\]

\[
(\hat{u} + u) e = \hat{u} e + \frac{e^2}{2}
\]

\[
\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right) = -\gamma
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0
\]

\[
\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2 - \hat{u}^2}{2}\right) = \gamma
\]
CFL # = \max_i \frac{U_i \Delta t}{\Delta X_i}
Nonlinear Scalar Conservation Laws

- Scalar conservation laws
- Finite volume for scalar conservation laws
- System of conservation law and FV scheme
- High resolution (2\textsuperscript{nd} order) schemes
- Solving 2D and 3D problems
Primitive, Conservative and Integral Forms

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f_x(u) + \frac{\partial}{\partial y} f_y(u) = g \]

2D, scalar

\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u) = g \]

\[ f = (f_x, f_y) \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{g} \]

N-D, system of conservation laws
\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u) = g \]

Conservative form

\[ \frac{\partial u}{\partial t} + \frac{df}{du} \cdot \nabla u = g \]

primitive form

[\int_{\Omega} \left( \frac{\partial u}{\partial t} + \nabla \cdot f(u) \right) d\mathbf{x} = \int_{\Omega} g d\mathbf{x}]

[\frac{d}{dt} \int_{\Omega} u d\mathbf{x} + \int_{\partial \Omega} \hat{n} \cdot f(u) d\mathbf{x} = \int_{\Omega} g d\mathbf{x}]

integral form
Examples of Conservation Laws

\[
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = 0 \quad f(u) = U \cdot U
\]

\[
\frac{\partial u}{\partial t} + U_x \frac{\partial u}{\partial x} + U_y \frac{\partial u}{\partial y} = 0 \quad f(u) = (U_x u - U_y u)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad f(u) = \frac{u^2}{2}
\]

Burgers Equation

Buckley-Leverett Equation
Burgers Eqn

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0
\]
Smooth Solution along Characteristic Lines

\[ x = a t + b \]

\[ u(\ a t + b, \ t) \]

does not depend on \( t \)

\[ 0 = \frac{d}{dt} u(at+b, t) = a \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \quad \text{Chain Rule} \]

\[ \text{Cons} \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \]

\[ \text{PDE} \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial u} \frac{\partial u}{\partial x} = 0 \]

What if \( a = \frac{d t}{d u} \)
Discontinuous Solution near SHOCKWAVES and its speed

\[ \frac{dX_c}{dt} : \text{speed of shock} \]

\[ X_c = X_c(0) + \frac{dX_c}{dt} \cdot t \]

\[ \int_a^b u \, dx = (X_c-a) \, u_L + (b-X_c) \, u_R \]

\[ \frac{d}{dt} \int_a^b u \, dx = \frac{dX_c}{dt} \, (u_L - u_R) \]

\[ f(u)|^b_a = f(u_R) - f(u_L) \]

\[ \Omega = [a, b] \]
\[
\frac{dx_c}{dt} (u_L - u_R) + f(u_R) - f(u_L) = 0
\]

\[
\frac{dx_c}{dt} = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{\Delta f(u)}{\Delta u}
\]

\[
f = \frac{u^2}{2}
\]
The Differential Form doesn’t work around shockwaves!

Use the integral form!