Finite Differences:
Convection-Diffusion Equation
**Introduction**

**General Problem**

\[ \mathbf{U} : \text{Velocity (given)}, \quad \nabla \cdot \mathbf{U} = 0 \]

\( u \): Temperature

\[ \mathbf{U} \cdot \nabla u = \kappa \nabla^2 u + f \quad \text{in} \quad \Omega \]

\( u \) prescribed on \( \Gamma \) (say)
Model Problem in $\mathbb{R}^1$

### Statement

**perforated wall: suction**

$$\mathbf{U} = -U \hat{x}$$

$$u = 1 \text{ at } x = 0$$

$$u = 0 \text{ at } x = \infty$$

$$\int_0^\infty f \, dx = 0$$

\[ -\kappa u_{xx} - U u_x = f \]

\[ -\varepsilon u_{xx} - u_x = f/U \quad \varepsilon = \kappa/U \]

\[ u(0) = 1, \quad u(\infty) = 0 \]

\[ u(0) = 1, \quad u(\infty) = 0 \]
Introduction

Model Problem in $\mathbb{R}^1$

Solution for $f = 0$...

$u = e^{-x/\varepsilon}$

Why?

$-\varepsilon \left( \frac{1}{\varepsilon^2} e^{-x/\varepsilon} \right) - \left( -\frac{1}{\varepsilon} e^{-x/\varepsilon} \right) = 0$
Introduction

Model Problem in $\mathbb{R}^1$

...Solution for $f = 0$

\[ e^{-x/\varepsilon} \]

- $\varepsilon = 0.1$
- $\varepsilon = 0.05$
- $\varepsilon = 0.025$
Singular Perturbation Theory (small $\varepsilon$)

Model Problem in $\mathbb{R}^1$

Solution for $f \neq 0$

- $-\varepsilon u_{xx}$
- $-u_x = f/U$

inner solution: boundary condition

$\varepsilon = 0$

outer solution:

$u \sim e^{-x/\varepsilon} + \frac{1}{U} \int x \, f \, dx'$

$u_{\text{unif}}$ $u_{\text{inner}}$ $u_{\text{outer}}$

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Finite Differences 5
1. Solution has structure / features — a boundary layer.
   How many grid points do we need?
   Where should they be placed?

2. Dissipative term \((-\varepsilon u_{xx})\) has small coefficient.
   How will this affect STABILITY?

3. Operator is NONSYMMETRIC.
Centered Differences

Difference Formulas

\[ \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} = v''(x_j) + \frac{\Delta x^2}{12}v^{(4)}(x_j + \theta \Delta x) \]

\[ \frac{v_{j+1} - v_{j-1}}{2\Delta x} = v'(x_j) + \frac{\Delta x^2}{6}v'''(x_j + \theta \Delta x) \]
Finite Differences

Solution

Centered Differences

Discrete equations \((f = 0)\)

\[-\varepsilon u_{xx} - u_x = 0 \quad u(0) = 1, \quad u(\infty) = 0\]

\[
\Rightarrow \begin{cases} 
-\varepsilon \frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{\Delta x^2} - \frac{\hat{u}_{j+1} - \hat{u}_{j-1}}{2\Delta x} = 0, \quad j = 1, \ldots \\
\hat{u}_0 = 1, \quad \hat{u}_j \to 0 \text{ as } j \to \infty
\end{cases}
\]
Centered Differences

Numerical Examples...

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{20} \]

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{40} \]
Centered Differences

...Numerical Examples

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{80} \]

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{160} \]
Define $P = \frac{\Delta x}{\varepsilon}$: numerical non-dimensional parameter.

Discrete equations:

$$-(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}) - \frac{P}{2}(\hat{u}_{j+1} - \hat{u}_{j-1}) = 0,$$

or

$$\left(1 + \frac{P}{2}\right)\hat{u}_{j+1} - 2\hat{u}_j + \left(1 - \frac{P}{2}\right)\hat{u}_{j-1} = 0.$$
Finite Differences Solution

Centered Differences

...Grid Peclet Number

\[ P \ll 1: \text{RESOLVED} \]

\[ P \gg 1: \text{UNRESOLVED} \]

\[ P = \Delta x / \varepsilon \]

\( \varepsilon \): boundary layer thickness
\[-\varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = \frac{u_{j+1} - u_{j-1}}{2\Delta x} = -\varepsilon u_{xx}(x_j) - u_x(x_j) + \tau_j \]

\[\tau_j = -\varepsilon \frac{\Delta x^2}{12} u^{(4)}(x_j + \theta \Delta x) - \frac{\Delta x^2}{6} u'''(x_j + \theta \Delta x) \]

\[\rightarrow 0 \text{ as } \Delta x \rightarrow 0 \Rightarrow \text{ Consistency} \]
Finite Differences Solution

Recall:

\[
\left(1 + \frac{P}{2}\right) \hat{u}_{j+1} - 2\hat{u}_j + \left(1 - \frac{P}{2}\right) \hat{u}_{j-1} = 0, \quad j = 1, \ldots
\]

\[
\hat{u}_0 = 1, \quad \hat{u}_j \to 0 \text{ as } j \to \infty \quad P = \frac{\Delta x}{\varepsilon}
\]

Assume:

\[
\hat{u}_j = \sum_{\ell=1}^{2} C_\ell \zeta_\ell^j, \quad j = 0, \ldots
\]
Plug in, $\zeta_1$, $\zeta_2$ are the two roots of the characteristic polynomial

$$
\left(1 + \frac{P}{2}\right) \zeta^2 - 2\zeta + \left(1 - \frac{P}{2}\right) = 0
$$

$$
\Rightarrow \quad \zeta_1 = \frac{1 - \frac{P}{2}}{1 + \frac{P}{2}}, \quad \zeta_2 = 1
$$

$$
\Rightarrow \quad \hat{u}_j = C_1 \left(1 - \frac{P}{2}\right)^j \left(1 + \frac{P}{2}\right) + C_2
$$
Finite Differences

Solution

Centered Differences

...Analytical Solution for \( \hat{u}_j \)

\[
\hat{u}_j = C_1 \left( \frac{1 - \frac{P}{2}}{1 + \frac{P}{2}} \right)^j + C_2
\]

Apply boundary conditions:

- \( \hat{u}_j \to 0 \) as \( j \to \infty \) \( \Rightarrow \) \( C_2 = 0 \)
- \( \hat{u}_0 = 1 \) \( \Rightarrow \) \( C_1 = 1 \)

\[
\Rightarrow \quad \hat{u}_j = \left( \frac{1 - \frac{P}{2}}{1 + \frac{P}{2}} \right)^j .
\]
Centered Differences

Inspection of $\hat{u}_j : P \ll 1$...

For $P \ll 1$,

$$\hat{u}_j \sim \left[ \left( 1 - \frac{P}{2} \right) \left( 1 - \frac{P}{2} + \frac{P^2}{4} - \frac{P^3}{8} + \cdots \right) \right]^j$$

$$\sim \left[ 1 - P + \frac{P^2}{2} - \frac{P^3}{4} + \cdots \right]^j$$

Furthermore,

$$u(x_j) = u_j = e^{-x_j/\varepsilon} = e^{-j \Delta x / \varepsilon} = (e^{-P})^j.$$
Finite Differences Solution

Centered Differences

...Inspection of $\hat{u}_j : P \ll 1$

$$|e_j| \sim |(e^{-P})^j - \left(1 - P + \frac{P^2}{2} - \frac{P^3}{4} + \ldots\right)^j|$$

$$= \left| 1 - \left[ e^P \left(1 - P + \frac{P^2}{2} - \frac{P^3}{4} + \ldots\right) \right]^j (e^{-P})^j \right|$$

$$\left| 1 - \left(1 - \frac{P^3}{12} + \ldots\right)^j \right| \sim \frac{P^3j}{12} + \ldots$$

$$= \frac{1}{12} \frac{\Delta x^2}{\varepsilon^3} \sum_{x_j} \Delta x e^{-x_j/\varepsilon} + \ldots \text{ as } \Delta x \to 0, j \to \infty, x_j \text{ fixed.}$$
Convergence (Rate):

At fixed $x = x_j$, and fixed $\varepsilon$,

$$|e_j| \sim \Delta x^2 \left( \frac{1}{12 \varepsilon^3} e^{-x_j/\varepsilon} \right)$$

as $\Delta x (P = \frac{\Delta x}{\varepsilon}) \to 0$; second-order scheme.
Accuracy (finite $\Delta x$):

$u$ is resolved:

$e_j$ is small: $|e_j| \sim \frac{1}{12} P^2 \frac{x_j e^{-x_j/\varepsilon}}{\varepsilon}$

$\Delta x \ll \varepsilon$

maximum of $e^{-1}$
Centered Differences

...$P \ll 1$: Conclusions

\[
R^* = \frac{12e}{P^2} |\hat{u} - u(x)|_{x=\varepsilon}
\]

\[
u(x) = e^{-x}
\]

\[
\hat{u}_j = \left(\frac{1 - P/2}{1 + P/2}\right)^j
\]

<table>
<thead>
<tr>
<th>$P$</th>
<th>$R^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.1269</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0281</td>
</tr>
<tr>
<td>0.25</td>
<td>1.0068</td>
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<tr>
<td>0.125</td>
<td>1.0017</td>
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<tr>
<td>0.0625</td>
<td>1.0004</td>
</tr>
<tr>
<td>0.0313</td>
<td>1.0001</td>
</tr>
</tbody>
</table>
As \( P \to \infty \), \( u_j = \left( \frac{1 - P/2}{1 + P/2} \right)^j \to (-1)^j \equiv S_j \)

(Sawtooth, \( 2\Delta x \) wave,...).

For large \( P \), \( u_j = S_j \times \text{[slow attenuation as } j \to \infty]\).
“Convergence”:

For fixed $x$,

$$|e_j| \sim O(1) \quad (\hat{u}_j \not\to u_j)$$

as $P \equiv \frac{\Delta x}{\varepsilon} \to \infty$.

Note: for fixed $x$, $\hat{u}_j$ does converge to $u_j$ in the classical sense of fixed $\varepsilon$, $\Delta x \to 0$. 

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Accuracy (fixed $\Delta x$):

$u_{\text{inner}}$ is not resolved:

$e_j$ is $O(1)$ ($\hat{u}_1 \approx -1$) near $x = 0$
\( u^{\text{outer}} (= 0) \) is resolved (generally, \( f \) and \( u^{\text{outer}} \) vary slowly):

\( e^j \) is order unity (\( |\hat{u}_j| \sim O(1) \)) for \( x \) large

\[ P \gg 1: \text{Conclusions} \]
Finite Differences Solution

Discrete Equations ($f = 0$)...

\[ U = -U \hat{x}, \quad U > 0 \]

\[ \Delta x \]

\[ x_0 \quad x_1 \quad x_j \quad x_{j+1} \quad \rightarrow \infty \]

\[
\frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} = v''(x_j) + \frac{\Delta x^2}{12} v^{(4)}(x_j + \theta \Delta x)
\]

\[
\frac{v_{j+1} - v_j}{\Delta x} = v'(x_j) + \frac{\Delta x}{2} v''(x_j + \theta \Delta x)
\]
Finite Differences
Solution

Upwind Difference Treatment

Discrete Equations \((f = 0)\)

\[-\varepsilon u_{xx} - u_x = 0 \quad u(0) = 1, \ u(\infty) = 0\]

\[\downarrow\]

\[-\varepsilon \frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{\Delta x^2} - \frac{\hat{u}_{j+1} - \hat{u}_j}{\Delta x} = 0 , \quad j = 1, \ldots\]

\[\hat{u}_0 = 1 , \quad \hat{u}_j \to 0 \text{ as } j \to \infty\]
Define \( P = \frac{\Delta x}{\varepsilon} \) as before, \( u^{\text{inner}} \) resolved if \( P < 1 \).

Then

\[-(\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}) - P(\hat{u}_{j+1} - \hat{u}_j) = 0,\]

or

\[(1 + P)\hat{u}_{j+1} - (2 + P)\hat{u}_j + \hat{u}_{j-1} = 0\]
\[ -\varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} - \frac{u_{j+1} - u_j}{\Delta x} = -\varepsilon u_{xx}(x_j) - u_x(x_j) + \tau_j \]

\[ \tau_j = -\varepsilon \frac{\Delta x^2}{12} u^{(4)}(x_j + \theta \Delta x) - \frac{\Delta x}{2} u''(x_j + \theta \Delta x) \]

\[ \Rightarrow 0 \text{ as } \Delta x \rightarrow 0 \Rightarrow \text{Consistency} \]
Recall:

\[(1 + P)\hat{u}_{j+1} - (2 + P)\hat{u}_j + \hat{u}_{j-1} = 0, \quad j = 1, \ldots \]
\[\hat{u}_0 = 1, \quad \hat{u}_j \to 0 \quad \text{as} \quad j \to \infty \quad P = \frac{\Delta x}{\varepsilon}\]

Assume:

\[\hat{u}_j = \sum_{\ell=1}^{2} C_\ell \xi_\ell^j, \quad j = 0, \ldots \]
Characteristic polynomial: \( \zeta_1, \zeta_2 \) satisfy

\[
(1 + P)\zeta^2 - (2 + P)\zeta + 1 = 0
\]

\[
\Rightarrow \quad \zeta_1 = \frac{1}{1 + P}, \quad \zeta_2 = 1
\]

\[
\Rightarrow \quad \hat{u}_j = C_1 \left( \frac{1}{1 + P} \right)^j + C_2
\]

\[
\Rightarrow \quad \text{(boundary conditions)} \quad \hat{u}_j = \left( \frac{1}{1 + P} \right)^j, \quad j = 0, \ldots
\]
For $P \ll 1$, 

$$\hat{u}_j \sim (1 - P + P^2 + \cdots)^j.$$ 

Furthermore, 

$$u(x_j) = u_j = e^{-x_j/\varepsilon} = e^{-j\Delta x/\varepsilon} = (e^{-P})^j.$$
Thus, for $e_j = u_j - \hat{u}_j$

$$|e_j| \sim |(e^{-P})^j - (1 - P + P^2 + \ldots)^j|$$

$$= \left| 1 - [e^P(1 - P + P^2 + \ldots)]^j \right| |(e^{-P})^j|$$

$$\left| 1 - (1 + \frac{P^2}{2} + \ldots)^j \right| \sim \frac{P^2 j}{2} + \ldots$$

$$= \frac{1}{2} \frac{\Delta x}{\varepsilon^2} \sum_{x_j}^j \Delta x e^{-x_j/\varepsilon} + \ldots \text{ as } \Delta x \to 0, j \to \infty, x_j \text{ fixed.}$$
Convergence (Rate):

At fixed \( x = x_j \), and fixed \( \varepsilon \),

\[
|e_j| \sim \Delta x \left( \frac{1}{2 \varepsilon^2} e^{-x_j/\varepsilon} \right)
\]

as \( \Delta x \left( P = \frac{\Delta x}{\varepsilon} \right) \to 0 \); first-order scheme.
Accuracy (finite $\Delta x$):

$u$ is resolved:

$e_j$ is small: $|e_j| \sim \frac{1}{2} \frac{P}{P^2}$ not $\frac{x_j e^{-x_j/\varepsilon}}{\varepsilon}$ maximum of $e^{-1}$

but $P \ll 1$
As $P \rightarrow \infty$, $u_j = \left( \frac{1}{1 + P} \right)^j \rightarrow \delta_{j0}$.

No oscillations for any $P$. 

Note: $\Delta x \ll \varepsilon$
Accuracy (fixed $\Delta x$):

$u^{\text{inner}}$ is not resolved:

$e_j$ is $O(1)$ near $x = 0$

(more precisely, error in derivative . . . )
\( u_{\text{outer}} (= 0) \) is resolved (generally, \( f \) and \( u_{\text{outer}} \) vary slowly):

\( e^j \) is small (here zero) for \( x \) large

WHY?
Finite Differences Solution

Upwind Difference Treatment

Numerical Examples...

- $\varepsilon = 1/200$
- $\Delta x = 1/20$
- $P = 10$

- $\varepsilon = 1/200$
- $\Delta x = 1/40$
- $P = 5$
Finite Differences Solution

Upwind Difference Treatment

...Numerical Examples...

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{80} \]
\[ P = 2.5 \]

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{160} \]
\[ P = 1.25 \]
Finite Differences Solution

Upwind Difference Treatment

...Numerical Examples

\[ \varepsilon = \frac{1}{200} \]
\[ \Delta x = \frac{1}{160} \]
\[ P = 1.25 \]

Upwind

Centered
GOOD: stability, no oscillations, and “accurate” outer solution — for “all $P$.”

BAD: first-order convergence rate.

UGLY: numerical scheme modifies significantly delicate (singular) part of equation.
High-Order Upwinding

Discrete Equations ($f = 0$)

\[ U = -U \hat{x}, \quad U > 0 \]

\[ \Delta x \]

\[ \rightarrow \infty \]

\[ \hat{u}_{j+1} - 2 \hat{u}_j + \hat{u}_{j-1} - \frac{1}{\Delta x^2} \frac{3}{2} \hat{u}_j + 2 \hat{u}_{j+1} - \frac{1}{2} \hat{u}_{j+2} = 0 \]

or

\[-(\hat{u}_{j+1} - 2 \hat{u}_j + \hat{u}_{j-1}) - P \left( -\frac{3}{2} \hat{u}_j + 2 \hat{u}_{j+1} - \frac{1}{2} \hat{u}_{j+2} \right) = 0 \]
High-Order Upwinding
Error Equation

\[ e_j = u(x_j) - \hat{u}_j = u_j - \hat{u}_j \]

\[-\varepsilon \frac{e_{j+1} - 2e_j + e_{j-1}}{\Delta x^2} - \frac{1}{\Delta x} \left( -\frac{3}{2}e_j + 2e_{j+1} - \frac{1}{2}e_{j+2} \right) = \tau_j\]

\[ \tau_j = -\varepsilon \frac{\Delta x^2}{12} u^{(4)}(x_j + \cdot) + \frac{\Delta x^2}{3} u'''(x_j + \cdot) + \frac{\Delta x^3}{4} u^{(4)}(x_j + \cdot) \]

\[ \rightarrow 0 \text{ as } \Delta x \rightarrow 0 \Rightarrow \text{Consistency} \]
Obtain differential equation for $\tilde{u}(x)$

\[-\varepsilon \tilde{u}_{xx} - \tilde{u}_x = \varepsilon \frac{\Delta x^2}{12} \tilde{u}_{xxxx} - \frac{\Delta x^2}{3} \tilde{u}''' - \frac{\Delta x^3}{4} \tilde{u}_{xxxx} + \cdots ,\]

or

\[\frac{\Delta x^3}{4} \tilde{u}_{xxxx} - \varepsilon \tilde{u}_{xx} - \tilde{u}_x = \varepsilon \frac{\Delta x^2}{12} \tilde{u}_{xxxx} - \frac{\Delta x^2}{3} \tilde{u}''' + \cdots \]

with $\tilde{u}(x_j) \to \hat{u}_j$ as $\Delta x \to 0$. 
For $P \ll 1$, show by solving difference equation analytically, that

$$|e_j| \sim O(\Delta x^2) \text{ as } \Delta x(P) \rightarrow 0, j \rightarrow \infty \text{ (fixed } x).$$

**Stability:** $|e_j| \approx O(|\tau_j|)$. 

E1

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For $P \gg 1$, show by solving difference equation analytically, that

$$\hat{u}_j \sim \left( \frac{2}{3P} \right)^j \text{ as } P \to \infty.$$ 

No oscillations; outer solution intact.
Finite Differences

Solution

High-Order Upwinding

Why Successful?

Equivalent Differential Equation:

\[
\frac{\Delta x^3}{4} \tilde{u}_{xxxx} - \varepsilon \tilde{u}_{xx} - \tilde{u}_x = \varepsilon \frac{\Delta x^2}{12} \tilde{u}_{xxxx} + \cdots
\]

Small numerical dissipation \((\Delta x^3)\): ensures accuracy

High derivative \((\tilde{u}_{xxxx})\): ensures stability

(independent of \(\varepsilon\)).

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