Iterative Methods:
Multigrid Techniques
Background

– Developed over the last 25 years — Brandt (1973) published first paper with practical results.

– Offers the possibility of solving a problem with work and storage proportional to the number of unknowns.

– Well developed for linear elliptic problems — application to other equations is still an active area of research.

Basic Principles

1. Multigrid is an iterative method → a good initial guess will reduce the number of iterations:

   to solve \( A_h u_h = f_h \) by iteration, we could take \( u^0_h \sim u_{2h} \), where \( A_{2h} u_{2h} = f_{2h} \) . . .

   but . . . the number of iterations needed to solve \( A_h u_h = f_h \) still \( O(n^2) \).

   \[ h = \frac{1}{n+1} \]
2. If after a few iterations, the error is smooth, we could solve for the error on a coarser mesh, e.g. $A_{2h} e_{2h} = r_{2h}$.

- Smooth functions can be represented on coarser grids;
- Coarse grid solutions are cheaper.
If the \textit{high frequency} components of the error decay faster than the \textit{low frequency} components, we say that the iterative method is a \textit{smoother}.
Is Jacobi a smoother? 

\[ \text{\ldots \rightarrow NO} \]
Basic Principles

Smoother

Under-Relaxed Jacobi...

\[ R_{\omega J} = \omega R_J + (1 - \omega) I \]

\[ \lambda^k(R_{\omega J}) = \omega \lambda^k(R_J) + (1 - \omega) = 1 - \omega(1 - \lambda^k(R_J)) \]

\[ \kappa = 1, \ldots, n \]

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Basic Principles

Smoother

...Under-Relaxed Jacobi

Iterations required to reduce an error mode by a factor of 100

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7
Basic Principles

Recall,

Is Gauss-Seidel a good smoother?
Basic Principles

Smoother

...Gauss-Seidel

Iterations required to reduce an \textit{A error mode} by a factor of 100

\[\text{...GS is a \textit{good smoother}.}\]
Given $w_h$ we obtain $w_{2h}$ by restriction.

$$w_{2h} = I_{2h}^h w_h$$

$I_{2h}^h$: restriction operator (matrix).

Simplest procedure is injection

$$w_{2h,i} = w_{h,2i} \text{ for } i = 1, \ldots, \frac{n-1}{2}$$
Basic Principles

Restriction

Intuitively,

\[ \begin{align*}
    h & \quad \text{GOOD} \\
    2h & \quad \text{BAD}
\end{align*} \]
Basic Principles

If we write

$$\mathbf{w}_h = \sum_{k=1}^{n} c_k \mathbf{v}^k$$

$v^k$: eigenvectors of $A$

Only the modes $k = 1, \ldots, \frac{n-1}{2}$ are “visible” by grid $2h$.

“visible” by grid $2h$

$1, 2, \ldots, \frac{n-1}{2}$

aliased

$\frac{n+1}{2}, \ldots, n - 1, n$
Basic Principles

Restriction

Aliasing

Mode $k > (n - 1)/2$ on grid $h$ becomes $(n - k)$ mode on grid $2h$.
Basic Principles

- Only low modes in $h$ can be represented well in $2h$.
- Low modes on $h$ become higher modes in $2h$.

<table>
<thead>
<tr>
<th>$k = 1$</th>
<th>$\frac{n - 1}{2}$</th>
<th>$\frac{n + 1}{2}$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td>grid $h$</td>
<td></td>
</tr>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td>$2h$</td>
<td></td>
</tr>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td>$4h$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
Given $w_{2h}$ we obtain $w_h$ by prolongation

$$w_h = I_{2h}^h w_{2h}$$

$I_{2h}^h$: prolongation operator (matrix).

Typically, we use interpolation.

$$i = 1, \ldots, \frac{n-1}{2}$$

$$w_{h,2i} = w_{2h,i}$$

$$w_{h,2i+1} = \frac{1}{2} (w_{2h,i} + w_{2h,i+1})$$
Basic Principles

Prolongation

\[ 2h \]

\[ h \]
Interpolation introduces high frequency errors.
Two Grid (Correction) Scheme

One cycle

\[ u_{h}^{r+1} \leftarrow MG(u_{h}^{r}, f_{h}) \]

- Relax \( \nu_{1} \) iterations of \( A_{h} u_{h} = f_{h} \) with initial guess \( u_{h}^{r} \rightarrow u_{h}^{r+1/3} \).

- Compute \( r_{h} = f - A_{h} u_{h}^{r+1/3} \), and restrict \( r_{2h} = I_{2h}^{h} r_{h} \).

- Solve \( A_{2h} e_{2h} = r_{2h} \) on \( 2h \).

- Prolongate \( e_{h} = I_{h}^{2h} e_{2h} \), and correct \( u_{h}^{r+2/3} = u_{h}^{r+1/3} + e_{h} \).

- Relax \( \nu_{2} \) iterations of \( A_{h} u_{h} = f_{h} \) with initial guess \( u_{h}^{r+2/3} \rightarrow u_{h}^{r+1} \).
We solve \( u(0) = u(1) = 0 \)

\[-u_{xx} = 25\pi^2 (\sin(5\pi x) + 9 \sin(15\pi x)) \,.
\]

Initial guess: \( u^0 = 0 \)

Solution: \( u = \sin(5\pi x) + \sin(15\pi x) \)

Two grid scheme: \( h = \frac{1}{32}, \quad 2h = \frac{1}{16} \)

Solve using under-relaxed Jacobi with \( \omega = \frac{2}{3} \)
Two Grid (Correction) Scheme

Initial condition

Example

![Graph showing solution and error comparison]
After $\nu_1 = 2$ iterations on the fine mesh
Two Grid (Correction) Scheme

Example

After coarse grid correction (4 iterations)
After $\nu_2 = 2$ post smoothing iterations (end of cycle 1)
Two Grid (Correction) Scheme

Example

After $\nu_1 = 2$ iterations
Two Grid (Correction) Scheme

Example

After coarse grid correction
Two Grid (Correction) Scheme

After $\nu_2 = 2$ iterations (end of cycle 2)
Two Grid (Correction) Scheme

Multigrid convergence vs. single grid

Example

- Weighted Jacobi, $\omega=2/3$
- 2level multigrid: $v_1=2$, $v_2=2$
Multiple Grids

**V-Cycle**

One cycle

\[ u_{h}^{r+1} \leftarrow VG_{h}(u_{h}^{r}, f_{h}) \]

- Relax \( \nu_{1} \) times on \( A_{h} u_{h} = f_{h} \) with initial guess \( u_{h}^{r} \rightarrow u_{h}^{r+1/3} \).

- If \( h \equiv \) coarsest grid, go to (SKIP)

\[ \begin{align*}
r_{2h} & \leftarrow I_{2h}^{h}(f_{h} - A_{h} u_{h}^{r+1/3}) \\
e_{2h} & \leftarrow VG_{2h}(0, r_{2h})
\end{align*} \]

- Correct \( u_{h}^{r+2/3} = u_{h}^{r+1/3} + I_{h}^{2h} e_{2h} \).

- (SKIP) Relax \( \nu_{2} \) times on \( A_{h} u_{h} = f_{h} \) with initial guess \( u_{h}^{r+2/3} \rightarrow u_{h}^{r+1} \).
Multiple Grids

\( V \)-Cycle

Schematically
Multiple Grids

$V$-Cycle

2D Example...

Solve

$$-(u_{xx} + u_{yy}) = 1, \quad \in \Omega \equiv \text{unit square}$$

$$u = 0 \quad \text{on the boundary}$$
Multiple Grids

\( \text{V-Cycle} \)

...2D Example...

Parameter dependence

\[
\log_{10} |r|_L^2 \quad \omega
\]

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Iterative Methods: Multigrid Techniques
Multiple Grids

Convergence as a function of grid levels (same fine mesh)

\[ \text{V-Cycle} \]

...2D Example...
Multiple Grids

V-Cycle

...2D Example

Convergence as a function of grid levels (same coarse mesh)
Multiple Grids

$W$-Cycles
Full Multigrid Scheme

Putting it all together...

Schematically
More Advanced Topics

- Anisotropic grids/equations.
- Algebraic multigrid.
- Convergence theory.
- How to deal with other operators.