18.409: An Algorithmist’s Toolkit
Lecture 11

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Administrivia

- Will post pset tonight
  - But may add to it later
  - Not due for a while, since we haven’t covered everything on it yet
Today

- Polar bodies and convex body norms
- Start Fritz John’s theorem
The Polar of a Polytope

- Suppose we have a (bounded) polytope \( C \subseteq \mathbb{R}^n \) that contains the origin.
- Can write it as set of points satisfying some inequalities:
  \[
  C = \{ x \mid a_i \cdot x \leq b_i, \ i = 1, \ldots, k \}
  \]
- Can scale inequalities s.t. WLOG all \( b_i = 1 \)
  \[
  C = \{ x \mid a_i \cdot x \leq 1, \ i = 1, \ldots, k \}
  \]
- Can form new polytope called the *polar of \( C \)* given by
  \[
  C^* = \text{conv}(a_1, \ldots, a_k)
  \]
- See blackboard for examples.
Some Properties of the Polar

- If $A \subseteq B$, what does this mean about $A^*$ and $B^*$?
  - See blackboard
- What is $(C^*)^*$?
  - I claim that $(C^*)^* = C$.
  - See blackboard for examples
  - We’ll prove a little bit later today, when we have some more machinery
- How does the polar behave under scaling?
  - See blackboard
- What about under a change of coordinates?
  - See blackboard
- If $C$ is origin-symmetric, what does this say about $C^*$?
- What does it mean if the polar is lower-dimensional?
- What about unbounded?
A More General Definition

- Would be nice if our definition worked for any convex body, not just for polytopes
- What should the polar of the unit sphere be?
- What the polar of a sphere of radius r?
- What about an ellipse?
- And an ellipsoid?
- The idea is that any convex body can be thought of as the intersection of a (possible infinite) set of half-spaces
  - Called “supporting hyperplanes”
  - So can think of polar as the convex hull of a possible infinite set of points, coming from all of the supporting hyperplanes
- **Definition:** The polar of a convex body C is given by

\[
C^* = \{ x \in \mathbb{R}^n \mid x \cdot c \leq 1 \ \forall c \in C \}
\]
More on Polarity

\[ C^* = \{ x \in \mathbb{R}^n \mid x \cdot c \leq 1 \ \forall c \in C \} \]

- Why does this agree with our earlier definition in the case of polytopes?
  \[ C = \{ x \mid a_i \cdot x \leq 1, \ i = 1, \ldots, k \} \]
  \[ C^* = \text{conv}(a_1, \ldots, a_k) \]

- By new definition, clear that \( a_i \in C^* \)

- Need to check two things: (see blackboard)
  1) Polar is convex (good to know in general too!)
  2) New def doesn’t add anything not in \( \text{conv}(a_i) \)

- Our new definition will help us see that \((C^*)^* = C\), but will need one more theorem first
Separating Hyperplanes

- **Definition:** Given a convex body $K \subseteq \mathbb{R}^n$ and a point $p$, a *separating hyperplane* is a hyperplane that has $K$ on one side and $p$ on the other.
  - I.e., You have some vector $\eta$ s.t.
    - $\eta \cdot x \leq 1$ for all $x \in K$, and
    - $\eta \cdot p \geq 1$
  - Or could have RHS $= 0$, or any other constant
  - Call it a “strongly separating hyperplane” if second inequality is strict

- **Theorem (Separating Hyperplane Theorem):** If $K$ is convex and $p \notin K$, there is a hyperplane that strongly separates them.
Polar of the Polar

- Sketch of why $K^{**} = K$:
- $K^* = \{p \mid k \cdot x \leq 1 \quad \forall k \in K\}$
- $K^{**} = \{y \mid p \cdot y \leq 1 \quad \forall p \in K^*\}$
- If $y \in K$, $p \in K^*$, $y \cdot p \leq 1$ by definition, for any such $p$
  - So $y \in K^{**}$, and thus $K \subseteq K^{**}$
- Suppose $y \in K^{**}$, i.e. $p \cdot y \leq 1 \quad \forall p \in K^*$, but that $y \not\in K$
- Let $\{v \cdot x = 1\}$ be a strongly separating hyperplane for $y$ and $K$
- $v \cdot k \leq 1 \quad \forall k \in K$, so $v \in K^*$
- $v \cdot y > 1$, with $v \in K^*$, which is a contradiction
Norms and Symmetric Convex Bodies

• Convex bodies we talk about from now on (for a while) will be origin symmetric
  ◦ I.e., $x \in C \Rightarrow -x \in C$

• Recall that a norm on $\mathbb{R}^n$ is a map $q: \mathbb{R}^n \to \mathbb{R}$ such that:
  1) $q(ax) = |a| \cdot q(x)$ for $a \in \mathbb{R}$ (homogeneity)
  2) $q(x + y) \leq q(x) + q(y)$ (triangle inequality)
  3) $q(x) \geq 0$ for all $x$ (nonnegativity)
    • Actually implied by 1 and 2
    • Why?
  4) $q(x) = 0$ if and only if $x = 0$ (positivity)

• There’s a deep connection between norms and convex bodies
  ◦ Makes convex geometry a powerful tool for functional analysis

• Let $B_q = \{x \in \mathbb{R}^n \mid q(x) \leq 1\}$
  ◦ The “unit ball” for $q$

**Claim:** $B_q$ is a convex body
  ◦ Why?
Norms and Symmetric Convex Bodies (cont.)

- Can we make the correspondence go the other way?
  - I.e., Given a convex body $C$, can we produce a norm in which $C$ is the unit ball?
- Obvious restriction: $C$ must be origin-symmetric
  - Why?
- **Definition:** The *Minkowski functional* of an origin-symmetric convex body $C$ is the map $p_C : \mathbb{R}^n \rightarrow \mathbb{R}$ by
  $$p_C(x) = \inf_{\lambda > 0} \{ x \in \lambda C \}$$
- Will sometimes denote this by $\| x \|_C$
- This is a norm
  - Need to check homogeneity, triangle ineq., nonneg.
From the definitions, get:
\[ C = \{ x : \| x \|_C \leq 1 \} \]

For any norm q, can define its dual by
\[ p^*(x) = \sup_{v \neq 0} \frac{v \cdot x}{p(v)} \]

Duality of norms ↔ polarity of convex bodies
- See blackboard

Important examples: \( l_p \) norms (p ≠ 1)
- \( \| x \|_p := (\sum_i x_i^p)^{1/p} \)
- \( B_p^n = \text{unit ball in } l_p \)
- \( (B_p^n)^* = B_q^n \), where \( \frac{1}{p} + \frac{1}{q} = 1 \)
  - I encourage you to check this!
  - See following slides for 1, 2, \( \infty \), and pictures
\[ p = 2 \]
\[ p = \infty \]
\[ p = \frac{1}{2} \]: not a norm, and not convex
\( p = 1/2 \)
\( p = 1 \)
\( p = 2 \)
\( p = 3 \)
\( p = 4 \)
\( p = 5 \)
\( p = 10 \)
\( p = 100 \)
\( p = \infty \)
\[
p=1 \\
p=\infty \\
p=4 \\
p=4/3 \\
p=2 \\
p=2 \\
p=5 \\
p=5/4 \\
p=3 \\
p=3/2 \\
p=10 \\
p=10/9
\]
FRITZ JOHN’S THEOREM
How Far Can C be from $B_2^n$?

- How well can we approximate any convex body $C$ by a Euclidean ball?
- What’s the right way to ask this?
- **Version 1:** What’s the least $t$ s.t.
  \[
  aB_2^n \subseteq C \subseteq t aB_2^n
  \]
- **Not a great definition:** Can get convex bodies, even in $\mathbb{R}^2$, where $t$ has to be arbitrarily large
  - For example?
- **Better version:** What’s the least $t$ s.t.
  \[
  E \subseteq C \subseteq tE
  \]
  for some ellipse
- See blackboard for examples
Banach-Mazur Distance

- Motivates general definition:
- **Definition:** The Banach-Mazur distance $d(K,L)$, for $K,L$ convex, is the least positive $d$ for which there's a linear image $L'$ of $L$ s.t. $L' \subseteq K \subseteq dL'$
  - See next slide for picture
  - Easy to see that this is symmetric
- **Note:** Distance is multiplicative not additive
  - E.g., $d(K,K) = 1$, not 0
  - Would need to take a log to get a metric
Banach-Mazur Distance (cont.)

Figure 7. Defining the distance between $K$ and $L$. 
Fritz John’s Theorem

- A couple versions/variants that we’ll see
- Sometimes also has Loewner’s name attached
- **Theorem:** For any n-dimensional, origin-symmetric convex body K,
  \[ d(K, B^n_2) \leq \sqrt{n} \]
- So exists some ellipse E s.t.
  \[ E \subseteq C \subseteq \sqrt{n}E \]
- Tight for cube
- If didn’t require symmetry, bound would be n
  - Tight for simplex
- **Phrased another way:** Can change coords s.t.
  \[ B^n_2 \subseteq K \subseteq \sqrt{n}B^n_2 \]
- Which ellipse E gives us this bound?
- **Answer:** The ellipse of maximal volume inside K
Fritz John’s Theorem (cont.)

- Let $M$ be symmetric, positive definite
- What is the volume of the ellipsoid
  \[ E = \{ x \mid x^T M x \leq 1 \}? \]
- Not too hard to see it equals $v_n / \det(M)$
- Easier (and equivalent!): choose a basis $e_1, \ldots, e_n$, let $x_i = \langle x, e_i \rangle$, and look only at ellipses of the form
  \[ E = \left\{ x \mid \sum_{i=1}^{n} \frac{x_i^2}{\alpha_i^2} \leq 1 \right\} \]
- Volume equals $v_n \prod \alpha_i$
- “Axis-aligned” ellipses
  - Need to be a little careful and choose right basis at right time
Fritz John’s Theorem (cont.)

- **Theorem [John]:** K contains a unique ellipsoid of maximal volume. It is $B_2^n$ iff:
  1) $B_2^n \subseteq K$
  2) There are unit vectors $u_1,\ldots,u_m$ on the boundary of K and positive $c_1,\ldots,c_m$ s.t.

\[
\sum_{i=1}^{m} c_i u_i = 0
\]

\[
\sum_{i=1}^{m} c_i \langle x_i, u_i \rangle^2 = |x|^2
\]

- Will show later how other theorem follows from this one
Fritz John’s Theorem (cont.)

\[ \sum_{i=1}^{m} c_i u_i = 0 \quad \sum_{i=1}^{m} c_i \langle x_i, u_i \rangle^2 = |x|^2 \]

- Conditions mean:
  - Sphere touches boundary in a lot of places
  - They can be weighted so center of mass is the origin and inertia tensor = \( \text{Id}_n \)
  - Don’t all lie “on one side of the sphere” or “near a proper vector subspace”

- Can rewrite second one as

\[ \sum_{i=1}^{m} c_i u_i u_i^T = I_n \]
Good

Violates first condition

Good

Violates second condition
Proof Idea for John’s Theorem

- Will prove on blackboard, but I think that some pictures provide intuition
- Need to prove two things:
  1) If have contact points as required by theorem, then $B_2^n$ is the unique ellipsoid of maximum volume contained in $K$
  2) If $B_2^n$ is such an ellipsoid, can find these points.
- We’ll start with 1)
Proof Idea for John’s Theorem 1)

\[ \sum_{i=1}^{m} c_i u_i = 0 \quad \quad \sum_{i=1}^{m} c_i \langle x, u_i \rangle^2 = |x|^2 \]

- Will show \( B_2^n \) is axis-aligned ellipse of biggest volume
  - I claim this is enough. Why?
- If I give you unit vectors \( u_i \) on \( \partial K \), what is the largest convex body consistent with this fact?
Proof Idea for John’s Theorem 1)

- In the polar, each contact point $u_i$ gives constraint $u_i \cdot x \leq 1$, so $u_i$ is in polar
- If $E \subseteq K$, we thus have $E^* \supseteq K^* \supseteq \{u_1, \ldots, u_m\}$
- So just need to show that any ellipsoid containing the contact points has volume $\geq 1$, with equality only for Euclidean ball
  - This will be pretty easy
- See board for proof
Proof Idea for John’s Theorem 2)

\[ \sum_{i=1}^{m} c_i u_i = 0 \quad \sum_{i=1}^{m} c_i \langle x_i, u_i \rangle^2 = |x|^2 \]

- Need to show we can find these \( c_i \) and \( u_i \)
- Taking trace of second condition gives \( \sum c_i = n \)
- For origin-symmetric bodies, first condition is unnecessary
  - Why?
- Can rewrite second condition as
  \[ \sum_{i=1}^{m} c_i u_i u_i^T = \text{Id}_n \]
- **Key Idea:** Think geometrically about the space of ellipses = space of matrices = \( \mathbb{R}^{n^2} \)
- Let \( U_i = u_i u_i^T \)
- The above condition thus says that \( \text{Id}_n / n \) is in convex hull of \( U_i \)
Proof Idea for John’s Theorem 2)

- Suppose \( \text{Id}_n/n \) is not in convex hull of the \( U_i \) (so we can’t find \( c_i \) and \( u_i \) to satisfy our condition)
- Means that we have a separating hyperplane *in the space of matrices*
- For matrices \( A \) and \( B \) of the same size, let
  \[
  A \bullet B = \sum_{i,j} A_{ij} B_{ij}
  \]
  - Dot product in \( \mathbb{R}^{n^2} \)
- This gives some matrix \( H \) s.t. \( A \bullet H \geq 1 \ \forall \ A \in \text{conv}(U_i) \) and \( (\text{Id}_n/n) \bullet H < 1 \).
- We’ll (after a little preprocessing) show that we can slightly modify the unit ball be nudging the identity in the direction of \( H \) and get better volume
- See blackboard
Proof of “Rounding” Result

- I claim that this version of John’s Theorem implies the existence of an ellipse \( E \) s.t.
  \[
  E \subseteq C \subseteq \sqrt{n}E
  \]

- See blackboard

- **Side note:** If you just want to prove this rounding result and don’t also need to characterize \( E \), there’s another, possibly simpler, proof.
  - Here’s the picture