Reduced Forms for Positive Definite Forms

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1 Preliminaries

Binary Quadratic Form: \( f(x, y) = ax^2 + bxy + cy^2 \)

Discriminant: \( \Delta = b^2 - 4ac \)

Definite Binary Quadratic Form: \( \Delta = -D < 0 \)

Positive-Definite Binary Quadratic Form: A definite form with \( a, c > 0 \)

2 Outline

Here, we will:

- Define reduced form
- Show that the number of reduced forms with fixed \( \Delta \) must be finite
- State (proof is up next) and set up the reduction theorem for PD Binary Quadratic Forms

3 Definitions and Results

A reduced form in the positive-definite case is any binary quadratic form \( ax^2 + bxy + cy^2 \) such that:

\[
|b| \leq a \leq c
\]

Note that using these definition alone, we can proof the following

**P1.** For a reduced PD binary quadratic form with \( \Delta = -D, D > 0 \), we have that \( |b| \leq \sqrt{\frac{D}{3}} \):

\[
\begin{align*}
|b| & \leq a, |b| \leq c, \ a, |b|, c > 0 \\
\Rightarrow |b|^2 & \leq ac \\
4b^2 & \leq 4ac \\
3b^2 + \Delta & \leq 0 \\
b^2 & \leq \frac{-\Delta}{3} \\
|b| & \leq \sqrt{\frac{D}{3}}
\end{align*}
\]

Using this, we can prove our second result:

**P2. The number of reduced forms with a discriminant \( \Delta = -D \) is finite.** Note that due to P1, we know that for a fixed discriminant, we have \(-\sqrt{D/3} \leq b \leq \sqrt{D/3}\), which limits us to a finite set of values for \( b \). Then, given one of these values, we have that \( \Delta = b^2 - 4ac \), so \( b^2 - \Delta = b^2 + D = 4ac \). Thus, \( ac \) is fully specified, so there are finitely many possible factorizations \((a, c)\) for each value of \( b \). Thus, there are a finite number of solutions \((a, b, c)\) to the constraints, each of which specifies a binary quadratic form.

Finally, we can set up the result to be proven soon:

**P3. Every PD binary quadratic form is equivalent to a reduced form of equal discriminant.**