PART 1: INDEFINITE FORMS AND INFINITE SOLUTIONS

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1. OVERALL GOAL

The goal of this lecture is to show that assuming solutions to a suitable Pell’s equation, if an indefinite BQF \( f(x, y) \) has one solution, \( f(x, y) = r \), then it has infinitely many solutions.

2. REMINDERS AND DEFINITIONS

- A binary quadratic form is indefinite if it has a positive discriminant, i.e. \( f(x, y) = ax^2 + bxy + cy^2 \) such that \( \Delta = b^2 - 4ac > 0 \)
- Pell’s Equation: For a fixed integer \( n > 0 \), \( x^2 - ny^2 = 1 \). Pell’s equation has infinitely many integer solutions \( (x, y) \) when \( n \) is not a square.

3. BASIC CASE - SHOWING INFINITELY MANY SOLUTIONS

Let’s start with two equations:

(1) \( x^2 - ny^2 = a \)
(2) \( x_0^2 - ny_0^2 = 1 \), where \( (x_0, y_0) \neq 0, \pm 1 \)

Now let’s expand these equations using basic algebra

(1) \( (x + y\sqrt{n})(x - y\sqrt{n}) = a \)
(2) \( (x_0 + y_0\sqrt{n})(x_0 - y_0\sqrt{n}) = 1 \)

And let’s multiply these Equations (1) and (2) together

(3) \( [(x + y\sqrt{n})(x_0 + y_0\sqrt{n})][(x - y\sqrt{n})(x_0 - y_0\sqrt{n})] = a \)

Now let’s expand \( (x + y\sqrt{n})(x_0 + y_0\sqrt{n}) = xx_0 + (xy_0\sqrt{n} + x_0y\sqrt{n}) + yy_0 = x_1 + y_1\sqrt{n} \)
And let’s expand \( (x - y\sqrt{n})(x_0 - y_0\sqrt{n}) = xx_0 - (xy_0\sqrt{n} + x_0y\sqrt{n}) + yy_0 = x_1 - y_1\sqrt{n} \)

Notice that Equation 3 becomes

(4) \( [(x + y\sqrt{n})(x_0 + y_0\sqrt{n})][(x - y\sqrt{n})(x_0 - y_0\sqrt{n})] = (x_1 + y_1\sqrt{n})(x_1 - y_1\sqrt{n}) = x_1^2 - ny_1^2 = a \), where \( x_1 = xx_0 + yy_0n \) and \( y_1 = xy_0 + x_0y \)

Notice that Equation 3 expanded using the new terms \( (x_1, y_1) \) becomes \( x_1^2 - ny_1^2 = a \). You can continue this process by multiplying the new equation for \( a \) (Equation 3) by Equation 2 to get \( \{x_2, x_3, ..., x_n\} \) and \( \{y_2, y_3, ..., y_n\} \). You’ll end up with an infinite set of solutions \( (x_n, y_n) \) to the original equation, \( (1) \ x^2 - ny^2 = a \).

Date: October 20, 2017.
4. Set-Up For Binary Quadratic Forms Proof

Let us begin with a binary quadratic form: $f(x, y) = ax^2 + bxy + cy^2 = r$ with $b^2 - 4ac > 0$.

In order to show that we can use a similar method as we did in the Basic Case, let’s get this into a familiar form, by multiplying the BQF by $4a$:

- $4a(ax^2 + bxy + cy^2) = 4ar$
- $4a^2x^2 + 4abxy + 4acy^2 = 4ar$
- $(2ax + by)^2 - (b^2y^2 + 4acy^2) = 4ar, \Delta = b^2 - 4ac$
- $(2ax + by)^2 - \Delta y^2 = 4ar$

To avoid confusion by using the same variables from the basic case, I’ll switch from using $(x, y)$ to using $(\alpha, \beta)$.

Now, we have a form that looks similar to what we used in the Basic Case.

$(2ax + by)^2 - \Delta y^2 \rightarrow \alpha_1^2 - \Delta \beta_1^2 = 4ar, \text{ by setting } \alpha_1 = (2ax + by) \text{ and } \beta_1 = y$

We can also switch from the form $\alpha_1^2 - \Delta \beta_1^2 = 4ar \rightarrow (2ax + by)^2 - \Delta y^2 \text{ by setting } x = \frac{\alpha_1 - b\beta_1}{2a}, \text{ and } y = \beta_1$

Now, we can see that if $\alpha_1$ and $\beta_1$ are solutions to $\alpha_1^2 - \Delta \beta_1^2 = 4ar$ and $\frac{\alpha_1 - b\beta_1}{2a} \in \mathbb{Z}$, then $(\frac{\alpha_1 - b\beta_1}{2a}, \beta_1)$ is a solution to $f(x, y) = ax^2 + bxy + cy^2 = r$.

The next part of the proof will show that if $(\alpha_1, \beta_1)$ is a solution to $\alpha_1^2 - \Delta \beta_1^2 = 4ar$ such that $\frac{\alpha_1 - b\beta_1}{2a} \in \mathbb{Z}$ and $(\alpha_0, \beta_0)$ is a solution to $\alpha_0^2 - \Delta \beta_0^2 = 1$, then $(\alpha_2, \beta_2)$ (another solution to $\alpha_1^2 - \Delta \beta_1^2 = 4ar$) has the property that $\frac{\alpha_2 - b\beta_2}{2a} \in \mathbb{Z}$, and you can continue to find $\{\alpha_3, \alpha_4, ..., \alpha_n\}$ and $\{\beta_3, \beta_4, ..., \beta_n\}$ (which are also solutions to $\alpha_1^2 - \Delta \beta_1^2 = 4ar$) such that $\frac{\alpha_n - b\beta_n}{2a} \in \mathbb{Z}$.