1 Introduction

One of the interests in studying binary quadratic forms is to understand the differences between definite and indefinite forms. Recall that a binary quadratic form $ax^2 + bxy + cy^2$ is definite if the discriminant $\Delta = b^2 - 4ac$ is negative. If the discriminant is positive, the form is called indefinite.

Today, we’ll prove a result that holds for definite forms over the integers, but not for indefinite forms. We will see later in the course that this particular property has deeper connections in algebraic number theory, but for now the proof we give is elementary.

2 Representing Integers

Let $ax^2 + bxy + cy^2$ be a binary quadratic form over the integers (i.e. with $a, b, c \in \mathbb{Z}$). We say that the form represents an integer $m$ if there exist integers $x$ and $y$ such that $ax^2 + bxy + cy^2 = m$. Note that the representation need not be unique. However, when the form is definite, there can only be finitely many such representations:

**Proposition 2.0.1.** A definite binary quadratic form $ax^2 + bxy + cy^2$ over the integers can represent an integer $m$ in only finitely many ways.

*Proof.* The key insight is to view the form as a quadratic in $x$, then to complete the square and combine some terms. To make things simpler, first multiply through by $a$:

$$a^2x^2 + abxy + acy^2 = am.$$ 

Now, complete the square in the first two terms:

$$\left(ax + \frac{by}{2}\right)^2 + acy^2 - \frac{b^2y^2}{4} = am.$$ 

Collect the two $y^2$ terms and multiply through by 4 to give:

$$(2ax + by)^2 - \Delta y^2 = 4am.$$ 

Notice that both terms on the left are nonnegative – this is where we use the assumption that $\Delta < 0$. So, any solution $(x, y)$ must have $y^2 \leq \frac{-4am}{\Delta}$. In particular, there can only be finitely many values of $y$ that could possibly satisfy this equation. For each possible value of $y$, $x$ can take on at most 2 values because the remaining equation is a single-variable quadratic equation in $x$. We conclude that there are only finitely many solutions $(x, y)$.

3 The Indefinite Case: Pell’s Equation

The situation for indefinite forms is different. In fact, for a given indefinite form, the existence of one representation for an integer $m$ implies infinitely many representations! We won’t prove this for general forms, but we can look at one example.

Consider the indefinite form $x^2 - 2y^2$ with discriminant $\Delta = 8 > 0$. It has some obvious representations by this form; for example $(x, y) = (1, 0)$ or $(3, 2)$. Here’s something a little less obvious: given any representation $(x, y)$, one has that $(x', y') = (3x + 4y, 2x + 3y)$ is another representation. This is easy to verify:

$$(3x + 4y)^2 - 2(2x + 3y)^2 = (9x^2 + 24xy + 16y^2) - 2(4x^2 + 12xy + 9y^2)$$

$$= x^2 - 2y^2.$$ 

This transformation implies the existence of infinitely many representations. If we start with a representation $(x, y)$ with $x, y > 0$, it gets mapped to a representation $(x', y')$ with $x' > x$ and $y' > y$. By repeatedly applying this transformation, we get an infinite set of distinct representations.
How does one find such a mapping? Suppose we factor our form as follows:

\[ x^2 - 2y^2 = (x + \sqrt{2}y) \cdot (x - \sqrt{2}y). \]

Suppose we multiply the first term in this factoring by its evaluation on one of our solutions, e.g. \((x, y) = (3, 2)\):

\[
(x + \sqrt{2}y) \cdot (3 + 2\sqrt{2}) = 3x + 2\sqrt{2}x + 4y + 3\sqrt{2}y \\
= (3x + 4y) + \sqrt{2}(2x + 3y) \\
= x' + \sqrt{2}y'.
\]

Similarly, we can repeat for the second term in the factoring:

\[
(x - \sqrt{2}y) \cdot (3 - 2\sqrt{2}) = 3x - 2\sqrt{2}x + 4y - 3\sqrt{2}y \\
= (3x + 4y) - \sqrt{2}(2x + 3y) \\
= x' - \sqrt{2}y'.
\]

Putting these two together, we find that:

\[
x'^2 - 2y'^2 = (x' + \sqrt{2}y') \cdot (x' - \sqrt{2}y') \\
= \left( (x + \sqrt{2}y) \cdot (3 + 2\sqrt{2}) \right) \cdot \left( (x - \sqrt{2}y) \cdot (3 - 2\sqrt{2}) \right) \\
= \left( (x + \sqrt{2}y) \cdot (x - \sqrt{2}y) \right) \cdot \left( (3 + 2\sqrt{2}) \cdot (3 - 2\sqrt{2}) \right) \\
= 1 \cdot 1 \\
= 1.
\]

Also notice that if we instead started with a representation \((x, y)\) satisfying \(x^2 - 2y^2 = k\) for some arbitrary fixed \(k\), then we would have instead arrived at \(x'^2 - 2y'^2 = k\).

For a fixed integer \(n > 0\), the equation \(x^2 - ny^2 = 1\) is known as Pell’s equation. One can show that Pell’s equation has infinitely many integer solutions \((x, y)\) when \(n\) is not a square. Perhaps surprisingly, the approach presented here generalizes: finding infinitely many representations by an indefinite form reduces to finding one representation, then finding an integer solution to Pell’s equation for some \(n\) that depends on the form. Later in the course, we’ll have a deeper understanding of why this happens when we reconsider these problems in the context of quadratic number fields.