1 Subspaces

Let $V$ be a vector space over a field $K$. We define a vector subspace $U$ in $V$ as a subset $U \subset V$, which satisfies the properties of a vector space, or specifically,

1. $\forall u_1, u_2 \in U, \ u_1 + u_2 \in U$.
2. $\forall u \in U, \ k \in K, \ ku \in U$.

2 Orthogonality

Let $v_1, v_2$ be vectors in a vector space $V$ over a field $K$. Suppose there exists a symmetric bilinear form $f$ over $V$. Then we say $v_1$ and $v_2$ are orthogonal with respect to the form $f$ if

$$f(v_1, v_2) = 0$$

2.1 Orthogonal Subspaces

Suppose we have a vector space $V$ over field $K$, and let $f$ be a symmetric bilinear form over $V$. Let $U_1, U_2$ be two subspaces of $V$. We say that $U_1$ and $U_2$ are orthogonal subspaces if $\forall u_1 \in U_1, u_2 \in U_2$,

$$f(u_1, u_2) = 0$$

That is, every vector in $U_1$ is orthogonal to every vector in $U_2$.

2.2 Orthogonal Complements

Suppose we have a vector space $V$ over field $K$, and let $f$ be a symmetric bilinear form over $V$. Let $U$ be a subspace of $V$. Define $U^0$, the orthogonal complement of $U$, as

$$\{ v \in V \mid \forall u \in U, f(u, v) = 0 \}$$

Proposition 1 $U^0$ is a subspace of $V$.

We check the two subspace criteria.
1. Suppose $u_1, u_2 \in U^0$. Then $\forall u \in U, f(u, u_1) = 0$ and $f(u, u_2) = 0$. Then by linearity of the form, we have that $\forall u \in U, f(u, u_1 + u_2) = 0 + 0 = 0$. So $u_1 + u_2 \in U^0$.

2. Suppose $u_0 \in U^0, k \in K$. Then $\forall u \in U, f(u, u_0) = 0$. By linearity of the form, $f(u, ku_0) = k(0) = 0$. Thus, $ku_0 \in U^0$.

**Corollary 1** $U$ and $U^0$ are orthogonal subspaces.

This follows by definition of $U^0$.

### 3 Orthogonality and the Radical

We return to the notion of a radical. We first note that $V$ is a subspace of $V$ (trivially). We can consider $V^0$, or the orthogonal complement of $V$. We define the kernel, or radical, of the space $V$ with respect to the form $f$ as $V^0$.

If the kernel only contains 0, then we say that $V$ is nondegenerate with respect to the form $f$.

We define the radical of a subspace $U$ as the the set of all $u \in U$ such that $u$ is orthogonal to every vector in $U$. We denote the radical by $\text{rad}$.

### 4 Some Propositions

Let $V$ be a vector space over $K$, and $f$ be a symmetric bilinear form over $V$. Suppose $V$ is nondegenerate with $f$. Let $U$ be a subspace of $V$. Then the following are true.

**Proposition 2** $\dim U + \dim U^0 = \dim V$

This is left as an exercise to the reader.

**Proposition 3** $U^{00} = U$

Proof: Consider $u \in U$. By definition of $U^0$, $u$ is orthogonal to every $u'$ in $U^0$. Therefore, $u \in U^{00}$. Thus, $U \in U^{00}$. By the previous proposition, we see that $U$ and $U^{00}$ are both vector spaces with the same dimension, and $U \in U^{00}$, so $U = U^{00}$.

**Proposition 4** $\text{rad}(U) = \text{rad}(U^0) = U \cap U^0$

We prove this by showing that both of the radicals are just $U \cap U^0$. Suppose we have $u \in \text{rad}(U)$. Then by definition of the radical, $u \in U$, and by definition of the radical, $u$ is orthogonal to everything in $U$, so $u \in U^0$. Therefore, $\text{rad}(U) \subset U \cap U^0$. Now suppose $u \in U \cap U^0$. Then $u$ is in $U$ and orthogonal to everything in $U$, since $u \in U^0$, so $u \in \text{rad}(U)$. So $U \cap U^0 \subset \text{rad}(U)$, so $\text{rad}(U) = U \cap U^0$. 

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Now suppose \( u \in \text{rad}(U^0) \). Then by definition of the radical, \( u \in U^0 \), and \( u \) is orthogonal to everything in \( U^0 \), so \( u \in U^{00} \), which by the previous proposition, is just \( U \). So \( u \in U \), \( u \in U^0 \), so \( u \in U \cap U^0 \), so \( \text{rad}(U^0) \subset U \cap U^0 \). Now suppose \( u \in U \cap U^0 \). By the previous proposition, \( U = U^{00} \), so \( u \in U \) and \( u \in U^{00} \). Therefore, \( u \in U^0 \), and \( u \) is orthogonal to everything in \( U^0 \) since \( u \) is in \( U^{00} \), so we have that \( u \in \text{rad}(U^0) \) by definition of the radical. So \( \text{rad}(U^0) \subset U \cap U^0 \). Therefore \( \text{rad}(U^0) = U \cap U^0 = \text{rad}(U) \), as desired.

5 Orthogonal Decomposition

Recall the definition of the direct sum from the previous presentation. Consider a vector space \( V \) over a field \( K \). Let \( U_1, U_2, \ldots, U_n \) be a set of subspaces of \( V \) such that the following is true

1. \( U_i \) and \( U_j \) are orthogonal subspaces, for \( i \neq j \).
2. \( V = U_1 \oplus U_2 \oplus \ldots \oplus U_n \).

Then we say \( U_1, U_2, \ldots, U_n \) forms an orthogonal decomposition of \( V \).