The action of $\text{SL}_2(\mathbb{Z})$ on the upper half plane

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1 Basic definitions

Definition 1.1. The complex upper half plane $\mathbb{H}$ is the subset of the complex plane given by

$$\mathbb{H} := \{ z = x + yi \in \mathbb{C} | y > 0 \}.$$ 

Definition 1.2. $\text{SL}_2(\mathbb{Z})$ is defined to be the set of $2 \times 2$ matrices with integer entries and determinant 1, i.e. it consists of matrices of the form

$$\text{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}.$$ 

2 The action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$

Definition 2.1. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}.$$ 

This is called the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Example 2.2. The matrix $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts on $z$ by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z = \frac{1 \cdot z + 1}{0 \cdot z + 1} = \frac{z + 1}{1}.$$ 

The matrix \( T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) acts on \( z \) by

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z = \begin{pmatrix} 0 \cdot z + 1 \\ -1 \cdot z + 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -z \end{pmatrix}.
\]

One can show that this action sends \( \mathbb{H} \) to itself, and that it commutes with matrix multiplication (so if you multiply two matrices and apply it to \( z \), you get the same answer as if you apply one matrix at a time).

**Proposition 2.3.**

(i) If \( \tau \in SL_2(\mathbb{Z}) \) and \( z \in \mathbb{H} \), then \( \tau z \in \mathbb{H} \).

(ii) If \( \tau_1, \tau_2 \in SL_2(\mathbb{Z}) \), and \( z \in \mathbb{H} \), then \( (\tau_1 \tau_2)z = \tau_1(\tau_2 z) \).

**Proof.**

(i) Let \( z = x + y i \) with \( y > 0 \). Then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{ax + ayi + b}{cx + cyi + d} = \frac{((ax + b) + ayi)((cx + d) - cyi)}{((cx + d) + cyi)((cx + d) - cyi)} = \frac{\text{real stuff} + \frac{ay(cx + d) - cy(ax + b)}{(cx + d)^2 + (cy)^2}i}{(cx + d)^2 + (cy)^2},
\]

Since sums of squares are positive, the imaginary part of this is still positive.

(ii) Let \( \tau_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \) for \( i = 1, 2 \). Then,

\[ \tau_2 z = \frac{a_2 z + b_2}{c_2 z + d_2}, \]

so

\[
\tau_1(\tau_2 z) = \frac{a_1 \cdot \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \cdot \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{a_1 a_2 z + a_1b_2 + b_1c_2 z + b_1d_2}{c_1a_2 z + c_1b_2 + d_1c_2 z + d_1d_2} = (\tau_1 \tau_2)z.
\]

It’s easy to see that these are the coefficients of the product matrix. \( \square \)
3 The fundamental domain

A fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is a subset $F \subseteq \mathbb{H}$ such that for any point $z \in \mathbb{H}$ there exists some $\tau \in \text{SL}_2(\mathbb{Z})$ such that $\tau z \in F$, and no two points in $F$ can be mapped to each other.

We want to find a fundamental domain for our action. Let

$$F := \left\{ |z| > 1, -\frac{1}{2} < \text{Re}(z) < \frac{1}{2} \right\} \cup \left\{ |z| \geq 1, \text{Re}(z) = -\frac{1}{2} \right\} \cup \{ |z| = 1, \text{Re}(z) \leq 0 \}.$$ 

Theorem 3.1. The set $F$ defined above is a fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Proof. We’ll show that every point can be mapped to an element of $F$. We won’t prove the uniqueness part.

Let $z \in \mathbb{H}$. We want to move it into $F$.

1. Pick $\tau$ so that $\tau z$ maximizes the imaginary part of $\tau z$. Why does this exist? We calculated above that for $z = x + yi$,

$$\text{Im} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right) = \frac{y}{(cx + d)^2 + (cy)^2}.$$ 

For any $k$, there are finitely many $(c, d)$ such that $(cx + d)^2 + (cy)^2 \leq k$, so $\text{Im}(\tau z)$ achieves a maximum.

2. Recall that the matrix $S$ in Example 2.2 shifts $z$ by $+1$, so its inverse shifts by $-1$. By repeatedly applying $S$ or $S^{-1}$, we can move $\tau z$ into the strip

$$\{-1/2 \leq \text{Re}(x) < 1/2\}.$$ 

Say it’s been moved to $z'$. Note that $z'$ has the same imaginary part as $\tau z$. We claim that $|z'| \geq 1$. Recall that the matrix $T$ sends $z \mapsto -1/z$. Thus, if $|z'| < 1$, then

$$\text{Im} \left( -\frac{1}{z'} \right) = -\text{Im} \left( \frac{z'}{z'^2} \right) = \frac{\text{Im}(z')}{|z'|^2} > \text{Im}(z') = \text{Im}(\tau z)$$

contradicting the maximality of the imaginary part of $\tau z$. 

$\square$