Properties of Bilinear Forms
(Discriminant, Radical, and Direct Sums)

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Let $V$ be a vector space over a field $K$, and let $B = [\vec{e}_1 \ e_2 \ e_3 \ \cdots \ e_n]$ be a basis for the vector space $V$. Thus for any vector $\vec{v} \in V$ we can define its coordinates to be $\vec{c}$ if

$$\vec{v} = B\vec{c} = \sum_{i=1}^{n} c_i \vec{e}_i = c_1 \vec{e}_1 + c_2 \vec{e}_2 + \cdots + c_n \vec{e}_n$$

Define $f(u, v)$ to be a bilinear form over $V$. Then we can define a matrix $A$ associated with the bilinear form $f$ such that $a_{ij} = f(e_i, e_j)$. Therefore

$$f(x, y) = f(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n, y_1 \vec{e}_1 + y_2 \vec{e}_2 + \cdots + y_n \vec{e}_n) = \sum_{i,j} x_i y_j f(\vec{e}_i, \vec{e}_j) = \sum_{i,j} x_i y_j a_{ij} = x^T A y$$

**Definition:** The **discriminant** of a bilinear form with respect to a fixed basis $B$ is simply the determinant of $A$, the associated matrix.

Consider what happens after a change in basis:

- $B' = BP$, where $P$ is the change-of-basis matrix
- $A' = P^T A P$
- $\implies \det(A') = \det(P^T) \det(A) \det(P) = \det(A) \det(P)^2$ because $\det(A^T) = \det(A)$.
- Therefore you can only change the discriminant of a bilinear form up to a square in the field $K$ you are working in by changing the basis.
**Definition:** The *radical* of a form $f(u,v)$ with respect to a fixed basis $B$ is equal to the set of all vectors $\alpha \in V$ such that either

- $f(\alpha, \beta) = 0 \ \forall \beta \in V$.
- $f(\gamma, \alpha) = 0 \ \forall \gamma \in V$

The first equation yields $\alpha^T A \beta = 0$, and since this is true for all $\beta$, we have $\alpha^T A = 0$. Similarly, the second equation yields $A \alpha = 0$. However, since $(A \alpha)^T = \alpha^T A^T = \alpha^T A$ because $A$ is a symmetric matrix, these two representations are equivalent. Thus the radical of $f(u,v)$ is the set of all vectors $v$ such that $Av = 0$. In essence, $\text{rad}(f) = \text{null space}(A)$.

As for the degree of the radical, consider the rank-nullity theorem, which states that

$$\text{Rank}(A) + \text{null space}(A) = n$$

Therefore we have two cases:

- $\text{Discr}(A) = 0 \implies |\text{rad}(f)| > 0$
- $\text{Discr}(A) \neq 0 \implies |\text{rad}(f)| = \{0\}$

**Definition:** Let $U, W$ be subspaces of $V$. Then $V$ is said to be the *direct sum* of $U$ and $W$ (written $V = U \bigoplus W$) if

- $V = U + W$ (This just means that for all $v \in V$, there exist $u \in U$ and $w \in W$ such that $v = u + w$.
- $U \cap W = \{0\}$

**Lemma:** If $U, W$ are subspaces of $V$, then $V = U \bigoplus W$ if and only if for all $v \in V$, there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.

**Proof:** By the definition, we have existence. What is left to show is uniqueness. Assume that there is more than one representation. Then $u_1 + w_1 = u_2 + w_2$. Moving terms to one side gives $u_1 - u_2 = w_2 - w_1$. But $u_1 - u_2 \in U$ and $w_2 - w_1 \in W$. And since $U \cap W = \{0\}$, this forces us to have $u_1 = u_2$ and $w_1 = w_2$, as desired.