Lecture 1: Review of manifolds

Def: An $n$-dimensional topological manifold is a Hausdorff topological space $X$ with countable base which is locally homeomorphic to $\mathbb{R}^n$ (i.e., $V \subseteq X$ is a neighborhood of $P \in U \subseteq X$ and a map $\varphi: U \to \mathbb{R}^n$ such that $\varphi(U) \subseteq \mathbb{R}^n$ is open and $\varphi: U \to \varphi(U)$ is a homeomorphism.

A pair $(U, \varphi)$ with $U \subseteq X$ open and $\varphi$ having the above properties is called a local chart. An atlas is a collection of local charts $(U_i, \varphi_i)$ such that $U_i \cap U_j = X$. The definition above implies that a manifold admits an atlas.

Suppose $(U, \varphi)$ and $(V, \psi)$ are local charts on $X$ such that $U \cap V \neq \emptyset$. In this case we have the transition map $\psi \circ \varphi^{-1}: \varphi(U \cap V) \to \psi(U \cap V)$.
It is clear that the transition map is continuous.

**Def.** An atlas \( \{(U_x, \varphi_x)\} \) on \( X \) is \( C^k \), \( 1 \leq k \leq \infty \), respectively analytic if all its transition maps are \( C^k \), resp. analytic. Also, \( (U_x, \varphi_x) \) is complex analytic if \( n = 2m \), so that \( \mathbb{R}^n \cong \mathbb{C}^m \), and the transition maps are complex analytic.

**Def.** Two atlases \( \{(U_x, \varphi_x)\} \) and \( \{(U_\beta, \varphi_\beta)\} \) are compatible if the transition maps between \( U_x \) and \( U_\beta \) are \( C^k \), respectively analytic when \( U_x \cap U_\beta \neq \emptyset \). This is clearly an equivalence relation.

**Def.** A \( C^k \), respectively real or complex analytic structure on \( X \) is a well-defined equivalence class of compatible \( C^k \) resp. real or complex analytic atlases on \( X \).

**Note:** \( C^\infty \)-structure is called smooth.
Ex. let \( X = S^1 = \mathbb{R} \cup \infty \). Let \( U_0 = \mathbb{R}, \ U_\infty = S^1 \setminus \{0\} = \mathbb{R} \setminus \{0\} \cup \infty \).

Let \( \psi_0 : U_0 \to \mathbb{R}, \ \psi_0(x) = x \),
\[ \psi_\infty : U_\infty \to \mathbb{R}, \ \psi_\infty(x) = \frac{1}{x}. \]

The transition map \( \mathbb{R} \setminus 0 \to \mathbb{R} \setminus 0 \) is \( x \mapsto \frac{1}{x} \).

It is real analytic so \( S^1 \) is a real analytic manifold.

Similarly, \( X = S^2 = \mathbb{C} \cup \infty \) is a complex analytic manifold (from now on, complex manifold, for short).

Now let \( p \in X \) and \( (p \in U, \psi) \) be a local chart such that \( \psi(p) = 0 \).

Such a chart will be called a coordinate chart around \( p \). In particular, we have the local coordinates \( x_1, \ldots, x_n : U \to \mathbb{R} \) such that \( x_i(p) = 0 \), and \( x_i(q) \) determine \( q \) if \( q \) is closer to \( p \).

Def. A regular function \( f : U \to \mathbb{R} \) on an open set \( U \subset X \) is a function such that \( f \circ \psi^{-1} : \psi(U \cup U_2) \to \mathbb{R} \)
is \( C^k \), resp. real or complex analytic (from now on we will just call it regular), for some (and then every) atlas \((U_\alpha, F_\alpha)\). In other words, \( f \) is regular if it is expressed as a regular function of local coordinates for every local coordinate system.

The space of regular functions on \( U \) is denoted \( O(U) \).

Assume from now on that \( k = \infty, RA = CA \).

Let \( P \in X \), and \( P \in U \subset X \) is a neighborhood of \( P \), and let \( (U, \phi) \) be a local chart. A linear map (over IR on C, depending on whether \( X \) is real or complex) \( \partial(U) \rightarrow \mathbb{R} \) is a \underline{derivation} at \( P \) if

\[ \partial(fg) = \partial(f)g(P) + f(P)\partial(g). \]

Let \( T_U \) be the space of derivations at \( P \). It is clear that it's independent on \( \phi \) but we claim that it's also independent on \( U \). Indeed, if \( \partial(U) \mid_{\partial(U)} \) induces an isomorphism \( T_U \rightarrow T_U \).
Indeed, it suffices to check this for $\mathbb{R}^n$, in which case it's clear as elements of $T_u$ are of the form $a_1 \frac{\partial f}{\partial x_1}(0) + \cdots + a_n \frac{\partial f}{\partial x_n}(0)$. To prove this, one can use Taylor's formula with remainder in integral form.

Def. The space $T_p$ is called the tangent space to $X$ at $p$ and denoted $T_pX$.

Def. A map $f : X \to Y$ of manifolds is regular if it is expressible by regular functions in local coordinates. $f$ is a diffeomorphism if $f$ is bijective and $f$ is also regular. If $f$ is regular, then we have its differential $df_p = f_* : T_pX \to T_Y$ whenever $f(p) = q$. Namely, (a linear).

$\forall u \in T_pX$ and $v \in O(U)$, $v(U \cap Y) \ni q$, we have $f_* (u)(v) = v(f \circ h)$. In particular, if $\gamma : (a, b) \to X$ is a regular parameterized curve then $\forall t \in (a, b), \quad \gamma'(t) \in T_{\gamma(t)}X$. 

is a tangent vector called the velocity vector of \( y \) at \( t \).

**Example.** Let \( f : X^n \to Y^m \) be a regular map, with \( m \leq n \). We say that \( f \) is a **submersion** if \( f_\ast : T_x X \to T_y Y \) is surjective for each \( p \in X \).

Let \( Q = f(p) \) for some \( p \in X \).

**Prop.** If \( f \) is a submersion then \( f(Q) \) has a natural structure of an \( n \)-dimensional manifold.

**Proof.** Let \( U \ni P \) be a neighborhood and \( \psi : U \to \mathbb{R}^{n+m} \) be a map such that

\[
(U, \psi) \text{ is a local chart, let } K = \ker(f_\ast) \oplus \mathbb{R}^n.
\]

Then \( K \) is \( n \)-dimensional. Let \( L \subset \mathbb{R}^{n+m} \) be any complement of \( K \), and \( \Pi : \mathbb{R}^{n+m} \to \mathbb{R}^n \) be the projection along \( L \). By the implicit function theorem, \( \Pi \circ f^{-1} : f^{-1}(Q)U \to \mathbb{R}^n \) is a local chart, at least if \( U \) is chosen sufficiently small and...
Def. $f : X \to Y$ is an immersion if $T_x f : T_x X \to T_{f(x)} Y$ is injective for all $x \in X$.

Ex. 1) $S^1 \to \mathbb{R}$

2) $\mathbb{R}^2 \to T^2 = \mathbb{R}^2 / \mathbb{Z}^2$

$t \to (t, \sqrt{2}t)$

Def. An immersion $f$ is an embedding if $f(X) \subseteq Y$ is a homeomorphism.

In this case, $f(X) \subseteq Y$ is called an embedded submanifold.

Ex. $\mathbb{C} \setminus \mathbb{Q} \subseteq \mathbb{C}$

more generally, any open set $U \subseteq X$ is an embedded submanifold.

Def. An embedding $f$ is closed if the image of $f$ is closed in $Y$.

Then $f(X)$ is a closed embedded submanifold of $Y$.

Ex: $\{(x,y) : x^2 + y^2 = 1\}$
Ex. If $f : X \to Y$ is a submersion then $f^{-1}(Q), Q \in Y$ is a closed embedded submanifold.

Finally, note that if $X, Y$ are manifolds then $X \times Y$ is also naturally a manifold.

Def. A Lie group of class $C^k, 1 \leq k \leq \infty$, real or complex analytic is a manifold $G$ of this class such that the multiplication map

$\mu : G \times G \to G$ is regular.

In this case left and right translations are different.

Prop. The map $i : G \to G, i(g) = g^{-1}$ is regular.

Proof. It suffices to show it's regular near $I$, the rest follows by translation. In local coordinates we have

$m_{ij}(x \cdot y) = x + y + \ldots$ (Taylor expansion).
\[ \partial u(x,y)/\partial y = \text{Id} \] and thus by implicit function theorem the equation \[ \mu(x,y) = 0 \] can be solved by a regular function \[ y = \varphi(x) \]. This function encodes the inversion map.

Remark. A \( C^0 \)-Lie group is a \( C^0 \)-manifold with a group structure such that \( \mu \) and \( e \) are continuous. The Gleason-Yamabe theorem (solving Hilbert's 5-th problem) says that any such group is actually an analytic Lie group (so the regularity class does not matter). So from now on we distinguish only real and complex Lie groups.

Def. A homomorphism of Lie groups is a group homomorphism which is also a regular map.
Def. An isomorphism of Lie groups is a homomorphism $f: G \rightarrow H$ which is an isomorphism of groups, and $f^{-1}$ is a homomorphism of Lie groups (i.e., regular). We will see later that the last condition is in fact redundant.

Example 1. $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$

$\mathbb{R}^n$: real Lie group
$\mathbb{C}^n$: complex Lie group.

2. $\mathbb{R}^\times, \mathbb{R}_+^\times, \mathbb{C}^\times$ with multiplication.

3. $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

We have $\mathbb{R}^\times \cong \mathbb{R}_+^\times \times \mathbb{Z}/2\mathbb{Z}$,
$\mathbb{C}^\times \cong \mathbb{R}_+^\times \times S^1$ (fig. form of complex number)
$(\mathbb{R}_+^\times) \xrightarrow{\log} (\mathbb{R}_+^\times \times S^1) \xrightarrow{\exp}$

4. $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$ - open sets in $\mathbb{R}^{n^2}$, $\mathbb{C}^{n^2}$. 

real, complex.
5. $SU(2) = \{ A \in GL_2(\mathbb{C}), AA^+ = I, \det A = 1 \}$

$A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), a, b, c, d \in \mathbb{C}, \quad A^+ = \left( \begin{array}{cc} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{array} \right)$

$AA^+ = \left( \begin{array}{cc} aa + bb & ac + bd \\ \overline{a}c + \overline{b}d & \overline{a}a + \overline{b}b \end{array} \right) = I$

$ac + bd = 0 \Rightarrow d = \lambda \overline{a}, \quad c = -\lambda \overline{b}$

$|a|^2 + |b|^2 = 1, \quad |\lambda|^2 \left( |a|^2 + |b|^2 \right) = 1$

$\Rightarrow |A| = 1$. Also $ad - bc = \lambda (a \overline{a} + b \overline{b}) = 1$

$\Rightarrow \lambda = 1$. Thus $A = \left( \begin{array}{cc} a & b \\ \overline{b} & \overline{a} \end{array} \right)$

$|a|^2 + |b|^2 = 1 \quad a = x + iy$

$b = z + it$

$\Rightarrow x^2 + y^2 + z^2 + t^2 = 1$, so

$SU(2) \cong S^3$

6. We will see that $SL_n(K), \quad O_n(K), SO_n(K), Sp_{2n}(K)$ etc

are Lie groups ($K = \mathbb{R}, \mathbb{C}$)