lecture 12.

Hydrogen atom.

The representation theory of $SU(2,0)$ is relevant to quantum mechanics of the hydrogen atom. Recall that the main equation of QM is the Schrödinger equation

$$-\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = H \psi$$

where $H$ is the Hamiltonian (suppressing the Planck constant).

So $\psi(x,t) = e^{iHt} \psi(x,0)$.

The Hamiltonian $H$ has the form

$$H = -\frac{1}{2} \Delta + U(x) \quad \Delta = \text{Laplace operator.}$$

Where $U(x)$ is the potential.

For the hydrogen atom $U(x) = -\frac{1}{r}$ (again ignoring the units). Writing $r = |x|$, we get

$$H = -\frac{1}{2} \Delta - \frac{1}{r}.$$
The solution $\psi$ has the form
\[ \sum_{n} c_{n} e^{i E_{n} t} \psi_{n}(x), \] where
\[ H \psi_{n} = E_{n} \psi_{n}. \]
The solutions of the stationary Schrödinger equation $H \psi = E \psi$ can be obtained by passing to spherical coordinates. Namely, write
\[ R^{3} \rightarrow R + \mathbb{S}^{2}, \]
\[ x \rightarrow (r, \mathbf{u}). \]
Then $\Delta = \Delta_{r} + \frac{1}{r^{2}} \Delta_{u}$, where
\[ \Delta_{r} = \frac{1}{2r^{2}} \frac{\partial}{\partial r} \left( r^{2} \frac{\partial}{\partial r} \right). \]
So we get
\[ \frac{1}{2} \frac{\partial^{2} \psi}{\partial r^{2}} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r} \psi + \frac{1}{2r^{2}} \Delta_{u} \psi = -E \psi. \]
This can be solved by separation of variables. Namely, if $\xi(x)$ satisfies $\Delta_{u} \xi = \lambda \xi$, then
we have a solution
\[ \psi = f(r) \frac{\hat{r}}{r} \] where \( f \) satisfies
\[ f'' + \frac{2}{r} f' + \frac{2}{r^2} f + \left( \frac{\lambda}{r^2} + 2E \right) f = 0. \]
So it remains to solve the equation \( \Delta \xi = \lambda \xi \) (and find the possible \( \lambda \)). Since the operator \( \Delta \) is invariant under rotations, it acts by a scalar on every irrep. Also we have
\[ S^2 = SU(2) \Big/ U(1), \] so \( L^2(S^2) = \bigoplus_{m=0}^{\infty} V_m \otimes V_m^* \), where \( V_m^* \) is the 0 weight space.
Thus \( L^2(S^2) = \bigoplus_{k=0}^{\infty} V_{2k} \).

The zero weight vectors are polynomials of the function of the angle \( 0 \leq \phi \leq \pi \) (latitude), or of \( \cos \phi \). Moreover, they are polynomials.
of degree $k$. Also, orthogonality implies that these polynomials are orthogonal on $[-1, 1]$ under the uniform measure. These are nothing but Legendre polynomials $P_n(x)$.

\[ \int_{-1}^{1} P_n(x) P_m(x) \, dx = \delta_{nm}. \]

Also, $\Delta u = \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} (\sin \phi \frac{\partial}{\partial \phi})$.

So if no dependence of $\theta$, we have $\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} (\sin \phi \frac{\partial}{\partial \phi}) = \frac{1}{\sin \phi} \frac{\partial}{\partial z} \left( (1 - z^2) \frac{\partial}{\partial z} \right)$.

So if $P$ is of degree $k$ and

\[ \frac{\partial}{\partial z} (1-z^2) \frac{\partial P}{\partial z} = \lambda P \]

then

\[ \lambda = -k(k+1) \]