Proposition: Let $G_1, G_2$ be Lie groups, with $G_1$ connected, and $\varphi: G_1 \to G_2$ a morphism. Then $\varphi$ is uniquely determined by $d\varphi: T_1 G_1 \to T_1 G_2$

Proof: We have shown that $\varphi(\exp(x)) = \exp(d\varphi(x))$ so $\varphi$ is determined on elements $\exp(x)$. But any element close to 1 is of this form.

The commutator. If $x, y \in G$ are small, then

$$\exp(x) \exp(y) = \exp(\mu(x, y)),$$

where $\mu: U \times U \to G$ where $U$ is a neighborhood of 0.

It is clear that

$$\mu(x, y) = x + y + \frac{\mu_2(x, y)}{2} + \cdots$$

where $\mu_2$ is a quadratic function.

It is clear $\mu_2(x, 0) = \mu_2(0, y) = 0$ since $\mu(x, 0) = x$, $\mu(0, x) = y$.

Hence $\mu_2$ is a bilinear
form $g \times g \rightarrow g$.

Also $M_2$ is skew-symmetric since $M_2(x,-x) = 0$ (as $P(x, x) = 0$).

Def. The commutator $[x, y] : g \otimes g \rightarrow g$

is defined by $[x, y] = 2M_2(x, y)$.

So

$$\exp(x) \exp(y) = \exp(x + y + \frac{1}{2} [x, y] + \ldots)$$

cubic and higher.

Example $G = GL_n(K)$

$$\exp(x) \exp(y) = (1 + x + \frac{x^2}{2} + \ldots)(1 + y + \frac{y^2}{2} + \ldots)$$

$$= 1 + x + y + \frac{x^2}{2} + xy + \frac{y^2}{2} + \ldots$$

$$\exp(x + y + M_2(x, y))$$

$$= 1 + x + y + M_2(x, y) + \frac{x^2}{2} + \frac{xy}{2} + \frac{y^2}{2} + \ldots$$

$$\Rightarrow M_2(x, y) = \frac{xy - yx}{2}$$

so $[x, y] = xy - yx$.

Cor. If $G \subseteq GL_n(K)$ is a Lie subgroup then $T_1 G$ is closed under $[x, y] = xy - yx$. 
Define $x\delta y$ as $x \cdot y$. Let $G$ be a group.

**Theorem.** The Jacobi identity holds.

Given $x, y, z \in G$, we have:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$ 

It is crucial to note that:

If $x^2 = z$, then $x^2y = x^2 - y$. Otherwise, assume $x^2 \neq y$.

**Proof.** This is immediate from the above.

The adjoint action preserves the commutator. That is, $g \cdot [x, y] = [g \cdot x, g \cdot y]$.

**Example:** Let $G$ be the group $\mathbb{Z}/2\mathbb{Z}$. Let $x \in G$, then $x^2 = 0$.

- $3$
The Jacobi identity is equivalent to the statement that
\[ [ad x, ad y] = ad [x, y] \quad \forall x, y \in g.\]

To prove this, we consider the homomorphism
\[ Ad: \mathfrak{g} \to \mathfrak{gl}(g) \]
given by the adjoint action.

**Lemma:** \( ad = Ad \)

1. \( Ad(\exp(x)) = \exp(\Ad x) : \mathfrak{g} \to \mathfrak{g} \)

**Proof:**
\[
\exp(x) \exp(y) \exp(-x) = \exp(z + [x, z] + \cdots)
\]
so \( \exp(x) \exp(-x) = z + [x, z] + \cdots \)

More generally, \( \exp(tx) = \exp(-tx) \)
and \( z \to \exp(t \ ad x) z \)
are two 1-par. subgroups with the same tangent vectors at 1, so they coincide.
Now compute
\[
\text{Ad}(\exp(x) \exp(y) \exp(-x) \exp(-y)) =
\]
modulo terms of degree 2 in \( x, y \).

On one hand it is \([x, y] z\).

On the other hand we have:
\[
2 \rightarrow z - [y, z] \rightarrow z - [y, z] - [x, z] + [x[y]]
\]
\[
\downarrow
\]
\[
z - [x, z] - [y[y]] + [x[y]]
\]
\[
\rightarrow z + [x[y]] - [y[y]]
\]
\[
\square
\]

\textbf{Def.} A vector space \( g \) with a skew symmetric operation \([\cdot, \cdot]_{g}\) satisfying the Jacobi identity is called a Lie algebra.

\textbf{Rem.} This def. makes sense over any field and even commut. ring \( R \).

But if \( \frac{1}{2} R \) then skew-symmetry should be replaced by \([x, x] = 0\).
Cor.: If $G$ is a Lie group then $\mathfrak{g} = T_1G$ has a canonical structure of a Lie algebra. If $\phi: G_1 \to G_2$ is a homom. of Lie groups then $\phi_*: \mathfrak{g}_1 \to \mathfrak{g}_2$ ($\mathfrak{g}_i = T_1G_i$) is a homom. of Lie algebras. So, we have from $(G_1, G_2) \to \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$, we mean if $G$, connected.

Prop.: Any subpace of $\mathfrak{g}_n(K)$ closed under $[,]$ is a Lie algebra.

Def.: Let $\mathfrak{g} \subseteq \mathfrak{g}$ is a Lie subalgebra if it is a subspace closed under commutator. Further, a Lie subalgebra $\mathfrak{g}$ is an ideal if $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$.

Prop.: Let $H \subseteq G$ be a Lie subgroup. Then $\text{Lie}(H) \subseteq \text{Lie}(G)$ is a Lie subalgebra.

2) $H \subseteq G$ is normal iff $\text{Lie} H \subseteq \text{Lie} G$ is an ideal if $H, G$ are connected (and $H \subseteq G$ normal $\Rightarrow \mathfrak{g} \subseteq \mathfrak{j}$ and ideal for any $H, G$).
Proof 1) \( x, y \in B \Rightarrow \exp(tx), \exp(uy) \in H \)
so \([x, y] \in B\), as
\[
[x, y] = \lim_{t, s \to 0} \log \left( \frac{e^{tx} e^{sy} - e^{-tx} e^{-sy}}{ts} \right).
\]

2) If \( x, y \in B \) small then
\[
e^{xy} e^{-x} = e^{xy} e^{-x} = e^{y} \cdot \frac{1}{e^{x} \cdot [x, y] + \frac{1}{2} [x, y]^2} + \cdots
\]
which is in \( H \). So \( G \) maps a small


neighbourhood of \( 1 \) in \( H \) into \( H \) (as \( B \) is generated by its small neighbourhood of \( 1 \)). Thus \( gHg^{-1} \in H \)

for any \( g \in C \).

Conversely, if \( gHg^{-1} \in H \) \( \forall g \in C \)

then \( \forall x, y \in C \), \( gxyg^{-1} \in C \) (as \( g \cdot x \cdot y \cdot g^{-1} = gxyg^{-1} \))

so taking \( g \in C \), and sending \( x \to 0 \), we get
\[
[x, y] \in B.
\]