Thm. Let $G$ be a Lie group acting on a manifold $M$, $m \in M$

(i) The stabilizer $H = G_m$ is a closed Lie subgroup, with Lie algebra $\mathfrak{g} = \{ x \in \mathfrak{g} \, | \, \text{for all } x \} \subset \mathfrak{g}$, where $\rho^*_x(x)$ is the vector field corresponding to $x$.

(ii) The map $G_m \to M$ given by $g \mapsto g \cdot m$ is an immersion so the orbit $O_m$ is an immersed submanifold with tangent space $T_m O = \mathfrak{g}$.

Proof. It suffices to show that for some neighborhood $U$ of $m \in G$, the intersection $U \cap G_m$ is an embedded submanifold with tangent space $T_m G_m = \mathfrak{g}$.

Note that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$, since commutator of vector fields vanishes at $m$. Vanishes at $m$.

Also for $h \in G$, $\exp(t \theta) \in G_m$. 


Now choose a complement \( u \) of \( y \) in \( g \), so \( g = y + u \). Then \( \phi_u : u \to T_{y,\mu} \) is injective. By the implicit function theorem, the map \( u \to M, y \to \phi_u(e^{y}) \) is injective for small \( y \in u \), so \( \exp(y) \in G_\mu \iff y = 0 \).

But in a small neighborhood \( U \) of \( \exp \), any element \( g \) can be uniquely written as \( e^{x} \), \( y \in u \), \( x \in h \), we see that \( ge^{x} \in G_{\mu} \iff g \in \exp(u) \). So since \( \exp(u) \) is a submanifold, we get that \( H \) is a submanifold.

Same philosophy shows that we have an isomorphism \( T_{y,\mu}(G_{\mu}) \cong \mathfrak{g} \), so \( \phi_u : u \to T_{y,\mu}M \) shows that \( G_{\mu} \to M \) is an immersion.

Cor 1. If \( f : G_1 \to G_2 \) is a morphism of Lie groups and \( f_x : G_1 \to G_2 \) the \( x \)-th
morphism of Lie algebras. Then \( k_{1f} \) is a closed Lie subgroup with Lie algebra \( k_{1f}^* \), and the map \( G / k_{1f} \rightarrow G_2 \) is an immersion. If \( \text{Im} f \) is a submanifold, and thus a closed Lie subgroup, we have an isomorphism \( \text{Im} f \cong G / k_{1f} \).

Let \( \text{Apply this to the } G \text{ action of } G \) on \( G_2 \) via \( g \cdot a = f(g) \cdot a \).

Cor. Let \( V \) be a rep. of a Lie group \( G \) and \( v \in V \). Then the stabilizer \( G_v \subset G \) is a closed Lie subgroup with Lie algebra \( g_v = \{ x \in G, x v = 0 \} \).

Ex. Let \( A \) be a f.d. associative algebra. Then the group \( \text{Aut}(A) \) is a Lie group with Lie algebra

\[ \text{Der } A = \{ d: A \rightarrow A \mid d(ab) = da \cdot b + a \cdot db \} \]

Indeed, if \( g(ab) = g(a)g(b) \) then taking log gives \( d(ab) = d(a)b + a \cdot d(b) \).
Same if \( A \) is a f.d. Lie algebra.

Def. The center of \( g \) is \( \{ x \in g \mid [x, g] = 0 \text{ for all } y \in g \} \).

Thm. Let \( G \) be a connected Lie group, then its center \( Z \) is a closed Lie subgroup with Lie algebra \( Z(g) \).

Proof. Let \( g \in G \), \( x \in g \). Then \( g \) commutes with \( \exp(tx) \) iff \( gx = xg \), i.e., \( \text{Ad}(g)x = x \). But for a connected Lie group, \( \text{Ad}(g)x \) generates \( G \), so if \( \text{Ad}(g)x = x \) for all \( x \in g \), then \( g \in Z(G) \). So \( Z(G) = \text{Ker Ad} \), \( \text{Ad} : G \to GL(g) \).

So the previous prop. implies the statement.

In general, \( G/K \) acts on \( Z(g) \), and \( Z(G) \) is a closed Lie subgroup with Lie algebra \( Z(g) \).

Def. \( G/Z(G) \) is called the adjoint group as it is the image of \( G \) in \( GL(g) \).
Campbell-Hausdorff formula.

Recall that the commutator of $g$ was obtained from the quadratic terms of $e^x e^y$. So one may wonder if by considering higher terms we may get more general operations on $g$. However, it turns out that the whole expansion of $e^{x+y}$ is entirely determined by the commutator.

**Thm.** For small $x,y$ enough $x,y \in g$, one has $e^{x+y} = e^{\mu(x,y)}$ where $\mu(x,y) = x + y + \sum_{n=2}^{\infty} \frac{\mu_n(x,y)}{n!}$, and $\mu_n(x,y)$ is a Lie polynomial of degree $n$ in $x$ and $y$. This series is convergent in some neighborhood of $(0,0)$ and universal (does not depend on $g$).

We will discuss the proof and an exact formula for $\mu_n(x,y)$ later.
Fundamental Theorems of Lie Theory

Thm 1. For a Lie group $G$, there is a bijection between connected Lie subgroups $H \leq G$ and Lie subalgebras $\mathfrak{h} \leq \mathfrak{g}$, given by $\mathfrak{h} = \text{Lie} H$.

Thm 2. If $G_1, G_2$ are Lie groups which are connected and simply connected, the map $\text{Hom}(G_1, G_2) \rightarrow \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$ is an isomorphism.

Thm 3. Any f.d. Lie algebra is the Lie algebra of a (real or complex) Lie group.

Corollary. The assignment $G \rightarrow \text{Lie} G$ is an equivalence between the category of connected simply conn. Lie groups and the category of f.d. Lie algebras.
Moreover, all connected Lie groups are of the form $G/Z(G)$; where $G/Z(G)$ is a discrete subgroup. Indeed, consider the functor $\text{Lie} : \text{SCLC} \to \text{FDLA}$ from simply connected Lie groups to f.d. Lie algebras. We know that this functor is essentially injective by Thm 3. Also, it is fully faithful by theorem 2.

We'll discuss the proofs later.

Complexification of Lie groups.

Let $\mathfrak{g}$ be a real Lie algebra. Then $\mathfrak{g}_c = \mathfrak{g} \otimes \mathbb{C}$ is a complex Lie algebra. We say that $\mathfrak{g}_c$ is the complexification of $\mathfrak{g}$ and $\mathfrak{g}$ is a real form of $\mathfrak{g}_c$.

Note that two non-isomorphic real Lie algebras can have the same
Complexification:
$m(n)_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C})$

but they are not isomorphic, for (as $\mathfrak{gl}(n,\mathbb{R})$ contains elements $X$ s.t. $\text{ad}(X)$ is not semisimple).

Def. let $G$ be a connected complex Lie group and $K \subset G$ is a Lie subgroup s.t. $\text{Lie}(K)$ is a real form of $\text{Lie}(G)$, then $K$ is called a real form of $G$.

Ex. $U(n)$ is a real form of $\text{GL}(n,\mathbb{C})$ and so is $\text{GL}(n,\mathbb{R})$ (but the latter is not connected).