Proof. Let \( A = A_s + A_n \) be the Jordan decomposition for \( A \). Then \( \text{ad} A = \text{ad} A_s + \text{ad} A_n \), and it is immediate to check that \( \text{ad} A_s , \text{ad} A_n \) commute.

Choose a basis in \( V \) such that in this basis, \( A_s \) is diagonal, \( A_n \) is strictly upper-triangular. Then it also gives a basis of matrix units \( E_{ij} \) in \( \text{End}(V) \). In this basis, the action of \( \text{ad} A_s \) is diagonal: \( \text{ad} A_s E_{ij} = (\lambda_i - \lambda_j) E_{ij} \), as is easily verified by a direct computation. Using this basis, it is also easy to check that \( \text{ad} A_n \) is nilpotent (see Exercise 5.7). Thus, \( \text{ad} A = \text{ad} A_s + \text{ad} A_n \) is the Jordan decomposition for \( \text{ad} A \), so \( (\text{ad} A)_s = \text{ad} A_s \).

By Theorem 5.59 applied to operator \( \text{ad} A \), we see that \( (\text{ad} A)_s \) can be written in the form \( P(\text{ad} A) \) for some polynomial \( P \in \mathbb{C}[I] \); moreover, since 0 is an eigenvalue of \( \text{ad} A \) (e.g., \( \text{ad} A A = 0 \)), we see that \( P(0) = 0 \). \( \square \)

Theorem 5.61. Let \( A \) be an operator \( V \to V \). Define \( \overline{A}_s \) to be the operator which has the same eigenspaces as \( A_s \) but complex conjugate eigenvalues: if \( A_s v = \lambda v \), then \( \overline{A}_s v = \overline{\lambda} v \). Then \( \text{ad} \overline{A}_s \) can be written in the form \( \text{ad} \overline{A}_s = Q(\text{ad} A) \) for some polynomial \( Q \in \mathbb{C}[I] \) (depending on \( A \)).

Proof. Let \( \{ v_i \} \) be a basis of eigenvectors for \( A_s \): \( A_s v_i = \lambda_i v_i \) so that \( \overline{A}_s v_i = \overline{\lambda}_i v_i \). Let \( E_{ij} \) be the corresponding basis in \( \text{End}(V) \); then, as discussed in the proof of Theorem 5.60, in this basis \( \text{ad} A_s \) is given by \( \text{ad} A_s E_{ij} = (\lambda_i - \lambda_j) E_{ij} \), and \( \text{ad} \overline{A}_s E_{ij} = (\overline{\lambda}_i - \overline{\lambda}_j) E_{ij} \).

Choose a polynomial \( f \in \mathbb{C}[I] \) such that \( f(\lambda_i - \lambda_j) = \overline{\lambda}_i - \overline{\lambda}_j \) (in particular, \( f(0) = 0 \)); such a polynomial exists by interpolation theorem. Then \( \text{ad} \overline{A}_s = f(\text{ad} A_s) = f(P(\text{ad} A)) \) where \( P \) is as in Theorem 5.60. \( \square \)

5.10. Exercises

(1) Let \( V \) be a representation of \( g \) and \( W \subset V \) be a subrepresentation. Then \( B_V = B_W + B_{V/W} \), where \( B_V \) is defined by (5.14).

(2) Let \( I \subset g \) be an ideal. Then the restriction of the Killing form of \( g \) to \( I \) coincides with the Killing form of \( I \).

5.2. Show that for \( g = \mathfrak{sl}(n, \mathbb{C}) \), the Killing form is given by \( K(x, y) = 2n \text{tr}(xy) \).

5.3. Let \( g \subset \mathfrak{gl}(n, \mathbb{C}) \) be the subspace consisting of block-triangular matrices:

\[
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\]

where \( A \) is a \( k \times k \) matrix, \( B \) is a \( k \times (n-k) \) matrix, and \( D \) is a \( (n-k) \times (n-k) \) matrix.

(1) Show that \( g \) is a Lie subalgebra (this is a special case of so-called parabolic subalgebras).

(2) Show that radical of \( g \) consists of matrices of the form \( \begin{pmatrix} \lambda \cdot I & B \\ 0 & \mu \cdot I \end{pmatrix} \), and describe \( g/\text{rad}(g) \).

5.4. Show that the bilinear form \( \text{tr}(xy) \) on \( \mathfrak{sp}(n, \mathbb{K}) \) is non-degenerate.

5.5. Let \( g \) be a real Lie algebra with a positive definite Killing form. Show that then \( g = 0 \). [Hint: \( g \subset \mathfrak{so}(g) \).]

5.6. Let \( g \) be a simple Lie algebra.

(1) Show that the invariant bilinear form is unique up to a factor. [Hint: use Exercise 4.5.]

(2) Show that \( g \cong g^* \) as representations of \( g \).

5.7. Let \( V \) be a finite-dimensional complex vector space and let \( A : V \to V \) be an upper-triangular operator. Let \( F^k \subset \text{End}(V) \), \(-n \leq k \leq n \) be the subspace spanned by matrix units \( E_{ij} \) with \( i - j \leq k \). Show that then \( \text{ad} A F^k \subset F^{k-1} \) and thus, \( \text{ad} A : \text{End}(V) \to \text{End}(V) \) is nilpotent.