7.11. Exercises

7.1. Let \( R \subset \mathbb{R}^n \) be given by

\[
R = \{ \pm e_i, \pm 2e_i \mid 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i, j \leq n, i \neq j \},
\]

where \( e_i \) is the standard basis in \( \mathbb{R}^n \). Show that \( R \) is a non-reduced root system. (This root system is usually denoted \( BC_n \).)

7.2. (1) Let \( R \subset E \) be a root system. Show that the set

\[
R^\vee = \{ \alpha^\vee \mid \alpha \in R \} \subset E^*.
\]

where \( \alpha^\vee \in E^* \) is the coroot defined by (7.4), is also a root system. It is usually called the dual root system of \( R \).

(2) Let \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \subset R \) be the set of simple roots. Show that the set \( \Pi^\vee = \{ \alpha_1^\vee, \ldots, \alpha_r^\vee \} \subset R^\vee \) is the set of simple roots of \( R^\vee \). [Note: this is not completely trivial, as \( \alpha \mapsto \alpha^\vee \) is not a linear map. Try using equation (7.17).]

7.3. Prove Lemma 7.17. (Hint: any linear dependence can be written in the form

\[
\sum_{i \in I} c_i \nu_i = \sum_{j \in J} c_j \nu_j,
\]

where \( I \cap J = \emptyset, c_i, c_j \geq 0 \). Show that if one denotes \( v = \sum_{i \in I} c_i \nu_i \), then \( (v, v) \leq 0 \).)

7.4. Show that \( \lvert P/Q \rvert = \lvert \det A \rvert \), where \( A \) is the Cartan matrix: \( a_{ij} = (\alpha_i^\vee, \alpha_j) \).

7.5. Compute explicitly the group \( P/Q \) for root systems \( A_n, D_n \).

7.6. Complete the gap in the proof of Theorem 7.37. Namely, assume that \( w = s_{i_1} \ldots s_{i_l} \). Let \( \beta_k = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}) \). Show that if we have \( \beta_k = \pm \beta_j \) for some \( j < k \) (thus, the path constructed in the proof of Theorem 7.37 crosses the hyperplane \( L_{\beta_j} \) twice), then \( w = s_{i_1} \ldots s_{i_j} \ldots s_{i_k} \ldots s_{i_l} \) (that means that the corresponding factor should be skipped) and thus, the original expression was not reduced.

7.7. Let \( w = s_{i_1} \ldots s_{i_l} \) be a reduced expression. Show that then

\[
[\alpha \in R_+ \mid w(\alpha) \in R_-] = \{ \beta_1, \ldots, \beta_l \}
\]

where \( \beta_k = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}) \) (cf. proof of Lemma 7.31).

7.8. Let \( \overline{C}_+ \) be the closure of the positive Weyl chamber, and \( \lambda \in \overline{C}_+, w \in W \) be such that \( w(\lambda) \in \overline{C}_+ \).

(1) Show that \( \lambda \in \overline{C}_+ \cap w^{-1}(\overline{C}_+) \).

(2) Let \( L_w \subset E \) be a root hyperplane which separates \( C_+ \) and \( w^{-1}C_+ \).

Show that then \( \lambda \in L_w \).

(3) Show that \( w(\lambda) = \lambda \).

Deduce from this that every \( W \)-orbit in \( E \) contains a unique element from \( \overline{C}_+ \).

7.9. Let \( w_0 \in W \) be the longest element in the Weyl group \( W \) as defined in Lemma 7.39. Show that then for any \( w \in W \), we have \( l(w w_0) = l(w_0 w) = l(w) - l(w) \).

7.10. Let \( W = S_n \) be the Weyl group of root system \( A_{n-1} \). Show that the longest element \( w_0 \in W \) is the permutation \( w_0 = (n \ldots 1) \).

7.11.

(1) Let \( R \) be a reduced root system of rank 2, with simple roots \( \alpha_1, \alpha_2 \). Show that the longest element in the corresponding Weyl group is

\[
w_0 = s_1 s_2 s_1 \ldots s_2 s_1 s_2 \ldots (m \text{ factors in each of the products})
\]

where \( m \) depends on the angle \( \varphi \) between \( \alpha_1, \alpha_2 \): \( \varphi = \pi - \frac{\pi}{m} \) (so \( m = 2 \) for \( A_1 \times A_1 \), \( m = 3 \) for \( A_2 \), \( m = 4 \) for \( B_2 \), \( m = 6 \) for \( G_2 \)). If you cannot think of any other proof, give a case-by-case proof.

(2) Show that the following relations hold in \( W \) (these are called Coxeter relations):

\[
s_i^2 = 1 \quad (s_i s_j)^{m_{ij}} = 1.
\]

(7.34)

where \( m_{ij} \) is determined by the angle between \( \alpha_i, \alpha_j \) in the same way as in the previous part. (It can be shown that the Coxeter relations is a defining set of relations for the Weyl group: \( W \) could be defined as the group generated by elements \( s_i \) subject to Coxeter relations. A proof of this fact can be found in [23] or in [31].)
7.12. Let \( \varphi : R_1 \tilde{\rightarrow} R_2 \) be an isomorphism between irreducible root systems. Show that then \( \varphi \) is a composition of an isometry and a scalar operator: 
\[
(\varphi(v), \varphi(w)) = c(v, w) \text{ for any } v, w \in E_1.
\]

7.13. (1) Let \( n_{\pm} \) be subalgebras in a semisimple complex Lie algebra defined by (7.25). Show that \( n_{\pm} \) are nilpotent.
(2) Let \( b = n_+ \oplus \mathfrak{h} \). Show that \( b \) is solvable.

7.14. (1) Show that if two vertices in a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same \( W \)-orbit.
(2) Show that for a reduced irreducible root system, the Weyl group acts transitively on the set of all roots of the same length.

7.15. Let \( R \subset E \) be an irreducible root system. Show that then \( E \) is an irreducible representation of the Weyl group \( W \).

7.16. Let \( G \) be a connected complex Lie group such that \( \mathfrak{g} = \text{Lie}(G) \) is semisimple. Fix a root decomposition of \( \mathfrak{g} \).
(1) Choose \( \alpha \in R \) and let \( i_\alpha : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g} \) be the embedding constructed in Lemma 6.42; by Theorem 3.41, this embedding can be lifted to a morphism \( i_\alpha : \text{SL}(2, \mathbb{C}) \rightarrow G \).
Let
\[
S_\alpha = i_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp \left( \frac{\pi}{2} (f_\alpha - e_\alpha) \right) \in G
\]
(c.f. Exercise 3.18). Show that \( \text{Ad} S_\alpha(h_\alpha) = -h_\alpha \) and that \( \text{Ad} S_\alpha(h) = h \) if \( h \in \mathfrak{h} \), \( (h, \alpha) = 0 \). Deduce from this that the action of \( S_\alpha \) on \( \mathfrak{g}^* \) preserves \( \mathfrak{h}^* \) and that restriction of \( \text{Ad} S_\alpha \) to \( \mathfrak{h}^* \) coincides with the reflection \( s_\alpha \).
(2) Show that the Weyl group \( W \) acts on \( \mathfrak{h}^* \) by inner automorphisms: for any \( w \in W \), there exists an element \( \tilde{w} \in G \) such that \( \text{Ad} \tilde{w}|_{\mathfrak{h}^*} = w \). [Note, however, that in general, \( \tilde{w}_1 \tilde{w}_2 \neq \tilde{w}_1 \tilde{w}_2 \).]

7.17. Let
\[
R = \{ \pm e_i \pm e_j \mid i \neq j \} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} \pm e_i \right\} \subset \mathbb{R}^8
\]
(in the first set, signs are chosen independently; in the second, even number of signs should be pluses). Prove that \( R \) is a reduced root system with Dynkin diagram \( \mathcal{E}_8 \).

8

Representations of semisimple Lie algebras

In this chapter, we study representations of complex semisimple Lie algebras. Recall that by results of Section 6.3, every finite-dimensional representation is completely reducible and thus can be written in the form \( V = \bigoplus n_i V_i \), where \( V_i \) are irreducible representations and \( n_i \in \mathbb{Z}_+ \) are the multiplicities. Thus, the study of representations reduces to classification of irreducible representations and finding a way to determine, for a given representation \( V \), the multiplicities \( n_i \). Both of these questions have a complete answer, which will be given below.

Throughout this chapter, \( \mathfrak{g} \) is a complex finite-dimensional semisimple Lie algebra. We fix a choice of a Cartan subalgebra and thus the root decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus n_i \mathfrak{g}_i \) (see Section 6.6). We will freely use notation from Chapter 7; in particular, we denote by \( \alpha_i, i = 1, \ldots, r \), simple roots, and by \( s_i \in W \) corresponding simple reflections. We will also choose a non-degenerate invariant symmetric bilinear form \( (\cdot, \cdot) \) on \( \mathfrak{g} \).

All representations considered in this chapter are complex and unless specified otherwise, finite-dimensional.

8.1. Weight decomposition and characters

As in the study of representations of \( \mathfrak{sl}(2, \mathbb{C}) \) (see Section 4.8), the key to the study of representations of \( \mathfrak{g} \) is decomposing the representation into the eigenspaces for the Cartan subalgebra.

**Definition 8.1.** Let \( V \) be a representation of \( \mathfrak{g} \). A vector \( v \in V \) is called a vector of weight \( \lambda \in \mathfrak{h}^* \) if, for any \( h \in \mathfrak{h} \), one has \( hv = (\lambda, h)v \). The space of all vectors of weight \( \lambda \) is called the weight space and denoted \( V[\lambda] \):

\[
V[\lambda] = \{ v \in V \mid hv = (\lambda, h)v \ \forall h \in \mathfrak{h} \}.
\]

(8.1)