Random Walks and Log Space

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Last Time

- Defined (d-regular) combinatorial and algebraic expanders
  - $(n,d,\rho)$-combinatorial expander if for all $S \subseteq V$ with $|S| \leq n/2$,
    - $|E(S, \overline{S})| \geq \rho d|S|
  - $(n,d,\epsilon)$-algebraic expander if eigenvalue gap is $\geq \epsilon$ (i.e., second largest eigenvalue is $\leq 1-\epsilon$)
- Equivalent when $\rho$ and $\epsilon$ are constant, because:
  - If $G$ is $(n,d,\epsilon)$-algebraic expander, then is also $(n,d,\epsilon^2)/2)$-combinatorial expander
  - If $(n,d,\frac{1}{2})$-combinatorial expander, then $\leq 1-\epsilon^2$/2
- Talked about spectral gap and how it relates to $S_{\text{mix}}$ for random walk to converge to stationary distribution
  - Random walk matrix $W = (1/d)A$
  - If prob vector is $\frac{1}{n}$, vector at next step is $W\frac{1}{n}$
- Uniform stationary distribution for regular graphs
- If start in distrib $p$, convergence to stationary bounded using spectral gap $\epsilon$: $||W^Tp - \pi||_2 \leq (1 - \epsilon)^t$
  - So within any $1/poly(n)$ in $O(\log n/\epsilon)$ steps

Some quick notes

- Gives easy proof that $\text{diam}(G) = O(\log n/\epsilon)$
  - Why?
- The following is a $(m^2, 8, \rho)$-expander:
  - $V = \mathbb{Z}_m^2$
  - Edges from $(x, y)$ to $(x \pm y, y)$, $(x \pm y+1, y)$, $(x, y \pm x)$, $(x, y \pm x+1)$
  - Useful, since can deal with very large version implicitly
- Can use to get much better error reduction than our pairwise independence construction
  - Not too hard, will do later
- Today want to talk about connection with log space computations

Undirected s-t Connectivity in RL

- Suppose have an undirected graph $G$, vertices $s, t$
- UPATH problem: is there a path from $s$ to $t$?
- Problem for directed graphs complete for NL
- Undirected problem complete for class $SL=\text{symmetric log space}$
  - Basically defined to be class undirected UPATH is complete for
- Claim: $SL \subseteq RL$
- Proof:
  - Take a random walk, starting at $s$
  - If exists path from $s$ to $t$, I claim you hit after $\text{poly}(n)$ steps with prob $\geq 2/3$ (or $1/poly$ if you want)
  - Why?
  - Two obvious questions:
    - Does this work for directed graphs and give NL=RL? [no]
    - Can we derandomize this and get SL=L? [yes!]
  - Proven by Reingold in 2005. Will give proof by Rozenman and Vadhan
- Note: Can show RL = PATH on graphs where this algorithm works
  - Seems like we might be able to strengthen proof of NL=L to get RL=L,
    but not known how to do this
Main Idea of the Proof

- For some graphs, can do in log space by trying all paths
  - Graphs with constant degree, log diameter vertices (original proof)
  - Graphs with poly degree and constant diameter (the one we’ll do today)
- Expanders have log diameter
  - And really really good expanders ($\epsilon = 1/2n^{1/2}$) have diam. 1
- So try to modify graph to improve expansion
- If have second largest eigenvalue $1-\epsilon$, squaring graph gives $(1-\epsilon)^2$
  - So squaring $O(\log n)$ times gives $(1-\epsilon)^{\text{poly}(n)}$, which will be $O(1)$ for big enough poly (because $\epsilon \geq 1/\text{poly}(n)$ as long as not bipartite

Problem: Degree blows up exponentially

Solution: Derandomized squaring
- Zig-zag product in original paper
- Idea is very powerful, will use again to prove PCP theorem

Derandomized Squaring

- Let $X$ be labeled $K$-regular graph with vertex set $[N]$
- Let $G$ be labeled $D$-regular graph with vertex set $[K]$
- Define KD-outregular graph $X \otimes G$ with:
  - Vertex set $[N]$
  - Edges are subset of paths in $X$ of length 2
  - Have edges $v(x)[y]$ if $y$ is neighbor of $x$ in $G$
  - I.e., Let $x, y \in [K]$ be an edge label in $X$, $a \in [D]$ be an edge label in $G$
  - Make edges from $v$ to $v(x)[x][a]$ for all $x, a$
- Not always in-regular
- If $X$ is consistently labeled, will be regular
- If $X, G$ consistently labeled, will be consistently labeled
- Will show that if $G$ is an expander, will improve connectivity almost as much as squaring, but only mult degree by a constant
- If $G$ constant degree, can do $O(\log(N))$ times and still have $\text{poly}(N)$ degree

Logspace Algorithm for UPATH

- Preprocessing: Show WLOG graph is 4-regular, consistently labeled, and has self-loop at each vertex (so nonbipartite)
- Perform derandomized squaring $O(\log N)$ times using constant degree expanders
  - Will increase spectral gap by a constant each step until get second eigenval to be constant
- Want eigenval to be $1/\text{poly}(n)$
  - If continue as above, can’t get past constant
  - Will use really nonconstant degree graphs at end, but not too many times, so total deg stays poly
- Will get graph with poly degree, diameter 1, so can just try all paths (=edges)

- Mild annoyance: will need to work with directed graphs
  - In general, very different, but will have nice ones
  - Will replace undir edges with 2 directed edges
- Digraph $X$ is $K$-outregular if outdegrees all = $K$
  - K-inregular if indegrees all = $K$
  - K-regular if K-outregular and K-inregular (all graphs today)
  - Self-loops count as both in and out edges
- When $K$-regular, still define walk matrix $W$ as adjacency matrix $K$, unif distrib. still stat.
- Define second biggest eigenval $(1-\epsilon)$ as sqrt. of second biggest eigenval of $W$ $W$
- Labeling: assign $K$ distinct #s in $[K]$ to outedges of each vertex
- Say consistent if all edges into each vertex have distinct labels
  - Can show every regular graph has consistent labeling, but won’t need this
  - Doesn’t exist for general digraphs, which is one of the parts that gives trouble generalizing to all of RL
**Main Technical Result**

- If $X$ is a consistently labeled $(N,K,\lambda)$-graph, $G$ is a $(K,D,\mu)$ graph, then $X \boxtimes G$ is an $(N, KD^2, f(\lambda, \mu))$ graph, with
  $$f(\lambda, \mu) = 1 - (1 - \lambda^2) \cdot (1 - \mu)$$

- Note that:
  - $f(\lambda, \mu) \leq \lambda + \mu$
  - $1 - f(1 - \gamma, 1/100) \geq (3/2)\gamma$ when $\gamma < 1/4$
  - When $\mu \to 0$ (G really good expander), $f(\lambda, \mu) \to \lambda^2$

- Proof will use:
  - **Lemma:** Let $A$=walk matrix of an $(N,D,\lambda)$ graph, $J_N = N \times N$ all-ones matrix/N. Then $A = (1 - \lambda)J_N + \lambda C$, where $||C|| \leq 1$
  - $||C|| = \max_{v \in \mathbb{R}^n} ||Cv|| / ||v||$

- Intuitively, means random step like going to unif. distrib with prob $(1 - \lambda)$, not getting further from unif with prob $\lambda$
  - Not exactly, b/c $C$ not necessarily stochastic, but right idea

- See board for proofs