I. Transmission Line Equations

A. Parallel Plate Transmission Line

Because $E$ must be perpendicular to electrodes and $H$ must be tangential,

$$E = E_x(z,t) \hat{i}_x$$

$$H = H_y(z,t) \hat{j}$$

Figure 8-1  The simplest transmission line consists of two parallel perfectly conducting plates a small distance $d$ apart.

$$\nabla \times \vec{E} = -\mu \frac{d\vec{H}}{dt} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t}$$

Figure 8-2  The potential difference measured between any two arbitrary points at different positions $z_1$ and $z_2$ on the transmission line is not unique—the line integral $L_x$ of the electric field is nonzero since the contour has magnetic flux passing through it. If the contour $L_2$ lies within a plane of constant $z$ such as at $z_4$, no magnetic flux passes through it so that the voltage difference between the two electrodes at the same value of $z$ is unique.

$$V(z,t) = \int_{z_1}^{z_2} \vec{E} \cdot dl = E_x(z,t) dz$$

$$I(z,t) = K_2(z,t) I = H_y(z,t) I$$
\[ \frac{di}{dt} = -L \frac{di}{dt} \quad L = \frac{\mu d}{w} \text{ henry/meter} \text{ Inductance per unit length} \]
\[ \frac{\partial \phi}{\partial t} = -C \frac{\partial \phi}{\partial t} \quad C = \frac{\varepsilon w}{d} \text{ farad/meter} \text{ Capacitance per unit length} \]

\[ L \frac{\partial i}{\partial t} \frac{\partial \phi}{\partial t} = \mu \varepsilon = \frac{1}{c} \]

**B. Coaxial Transmission Line**

Figure 8.3  Various types of simple transmission lines.

**EXAMPLE 8.1 THE COAXIAL TRANSMISSION LINE**

Consider the coaxial transmission line shown in Figure 8.3 composed of two perfectly conducting concentric cylinders of radii \( a \) and \( b \) enclosing a linear medium with permittivity \( \varepsilon \) and permeability \( \mu \). We solve for the transverse dependence of the fields as if the problem were static, independent of time. If the voltage difference between cylinders is \( v \) with the inner cylinder carrying a total current \( i \) the static fields are

\[ E_r = \frac{v}{r \ln (b/a)}, \quad H_\phi = \frac{i}{2\pi r} \]

The surface charge per unit length \( q \) and magnetic flux per unit length \( \lambda \) are

\[ q = \varepsilon E_r (r = a) \frac{2\pi v}{\ln (b/a)} \]

\[ \lambda = \int_a^b \mu H_\phi \, dr = \frac{\mu i}{2\pi} \frac{\ln b}{a} \]

so that the capacitance and inductance per unit length of this structure are

\[ C = \frac{q}{v} = \frac{2\pi \varepsilon}{\ln (b/a)}, \quad L = \frac{\lambda}{i} = \frac{\mu}{2\pi} \frac{\ln b}{a} \]

where we note that as required

\[ LC = \frac{\varepsilon \mu}{i} \]

Substituting \( E_r \) and \( H_\phi \) into (12) yields the following transmission line equations:

\[ \frac{\partial E_r}{\partial t} = -\frac{\partial H_\phi}{\partial z}, \quad \frac{\partial H_\phi}{\partial t} = -L \frac{\partial i}{\partial t} \]

\[ \frac{\partial E_r}{\partial z} = -\frac{\partial H_\phi}{\partial t}, \quad \frac{\partial H_\phi}{\partial z} = -C \frac{\partial v}{\partial t} \]

**C. Distributed Circuit Representation with Losses**

Figure 8.5  Distributed circuit model of a transmission line including small series and shunt resistive losses.
\[ i(z,t) - i(z+\Delta z,t) = C \Delta z \frac{\partial i(z,t)}{\partial t} + G \Delta z u(z,t) \]

\[ u(z,t) - u(z+\Delta z,t) = L \Delta z \frac{\partial i(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t) R \Delta z \]

\[ \lim_{\Delta z \to 0} \frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = \frac{\partial i}{\partial z} = -C \frac{\partial u}{\partial t} - Gu \]

\[ \lim_{\Delta z \to 0} \frac{u(z+\Delta z,t) - u(z,t)}{\Delta z} = \frac{\partial u}{\partial z} = -L \frac{\partial i}{\partial t} - iR \]

\[ R = \text{series resistance per unit length, ohms/meter} \]

\[ G = \text{shunt conductance per unit length, siemens/meter} \]

If lossless \((R=0, G=0)\):

**Telegrapher's equations**:

\[ \frac{\partial i}{\partial z} = -C \frac{\partial u}{\partial t} \]

\[ \frac{\partial u}{\partial z} = -L \frac{\partial i}{\partial t} \]

Including loss, **Poynting's theorem for the current**

\[ \text{equivalent form:} \]

\[ \nabla \cdot \frac{\partial}{\partial z} \left( \begin{array}{c} i \\ i \end{array} \right) = -C \frac{\partial u}{\partial t} - Gu \]

\[ \nabla \cdot \frac{\partial}{\partial z} \left( \begin{array}{c} u \\ i \end{array} \right) = -L \frac{\partial i}{\partial t} - iR \]

**Add**:

\[ u \frac{\partial i}{\partial z} + i \frac{\partial u}{\partial z} = \nabla \cdot (\nabla u) = -\frac{\partial}{\partial t} \left[ \frac{1}{2} C u^2 + \frac{1}{2} Li^2 \right] - Gu^2 - i^2R \]
D. Wave Equation (Lossless, R=0, G=0)
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = - \frac{C^2 \partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = - \frac{1}{L} \frac{\partial^2 u}{\partial t^2}
\]
\[
\frac{\partial^2 u}{\partial x^2} = - \frac{C^2 \partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{L} \frac{\partial^2 u}{\partial t^2}
\]

\[\text{Wave Equation}\]

II. Sinusoidal Steady State
A. Complex Amplitude Notation
\[
v(x,t) = Re \left[ \hat{v}(z) e^{j\omega t} \right]
\]
\[
i(x,t) = Re \left[ \hat{i}(z) e^{j\omega t} \right]
\]

Substitute into wave equation:
\[
\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{\omega^2}{c^2} \hat{v}(z), \text{ let } k = \frac{\omega}{c}
\]
\[
\frac{\partial^2 \hat{v}}{\partial z^2} + jk^2 \hat{v} = 0 \Rightarrow \hat{v}(z) = \hat{V}_+ e^{-jklz} + \hat{V}_- e^{+jklz}
\]
\[
\frac{d^2 \hat{v}}{dz^2} = -Lj\omega \hat{i} \Rightarrow \hat{i}(z) = \frac{1}{Lj\omega} \left( -j\delta \hat{V}_+ e^{-jklz} + \hat{V}_- e^{+jklz} \right)
\]
\[
\frac{\omega}{c} = \sqrt{rac{L}{C}} \Rightarrow \frac{k}{\omega} = \sqrt{\frac{L}{C}} = \sqrt{\frac{C}{L}} = z_0 \quad \text{Line Admittance}
\]
\[
z_0 = \frac{1}{z_0} = \sqrt{\frac{L}{C}} \quad \text{Line Impedance}
\]
\[
\hat{v}(z) = \hat{V}_0 \left( \hat{V}_+ e^{-jklz} - \hat{V}_- e^{+jklz} \right)
\]
\[
\hat{i}(z) = \hat{V}_0 \left( \hat{V}_+ e^{-jklz} + \hat{V}_- e^{+jklz} \right)
\]
\[ v(z,t) = \text{Re} \left[ \hat{v}_+ e^{i(\omega t-kz)} + \hat{v}_- e^{i(\omega t+kz)} \right] \]
\[ i(z,t) = \text{Re} \, V_0 \left[ \hat{v}_+ e^{i(\omega t-kz)} - \hat{v}_- e^{i(\omega t+kz)} \right] \]
\[ k = \frac{\omega}{c} = \omega \sqrt{LC} = \omega \sqrt{\mu} \]

B. Short Circuited Line \((v(r=0, t) = 0), v(r=l, t) = V_0 \cos \omega t\)

\[ \begin{align*}
\theta(s) &= \frac{V_0 \sin ks}{\sin k \lambda} \\
\hat{\theta}(s) &= -jV_0 \frac{\cos ks}{\sin k \lambda} \\
\lim_{k \lambda \to 1} \theta(s) &= \frac{jL \lambda \omega}{k \lambda} \\
\lim_{k \lambda \to 1} \hat{\theta}(s) &= -jV_0 \frac{1}{(L \lambda) \omega} \\
\end{align*} \]

Figure 8-15 The voltage and current distributions on a (a) short circuited and (b) open circuited transmission line excited by sinusoidal voltage sources at \(z = -l\). If the lines are much shorter than a wavelength, they act like reactive circuit elements. (c) As the frequency is raised, the impedance reflected back as a function of \(z\) can look capacitive or inductive making the transition through open or short circuits.

\[ v(z) = \hat{v}_+ e^{-jkl} + \hat{v}_- e^{jkl} \Rightarrow \hat{v}(z=0) = 0 = \hat{v}_+ + \hat{v}_- \Rightarrow \hat{v}_+ = -\hat{v}_- \]
\[ v(z=-l) = V_0 = \hat{v}_+ e^{jkl} + \hat{v}_- e^{-jkl} = \hat{v}_+ (e^{jkl} - e^{-jkl}) = 2j \sin k \lambda \hat{v}_+ \]
\[ \hat{v}_+ = -\hat{v}_- = \frac{V_0}{2j \sin k \lambda} \]
\[
\mathbf{f}(z) = \frac{V_0}{zj \sin \theta k} \left( e^{-j \theta k z} - e^{+j \theta k z} \right) = \frac{V_0 (e^{-j \theta k z} - e^{+j \theta k z})}{zj \sin \theta k} = -\frac{V_0 \sin \theta k z}{zj \sin \theta k} = \frac{zj V_0 \cos \theta k z}{zj \sin \theta k} = -\frac{j V_0 \cos \theta k z}{zj \sin \theta k}
\]

\[
\mathbf{i}(z) = V_0 \left( e^{-j \theta k z} - e^{+j \theta k z} \right) = \frac{V_0}{zj \sin \theta k} \left( e^{-j \theta k z} + e^{+j \theta k z} \right) = \frac{V_0}{zj \sin \theta k} \left( 2 \cos \theta k z \right) = \frac{2 V_0 \cos \theta k z}{zj \sin \theta k}
\]

\[
\mathbf{v}(z,t) = \text{Re} \left[ \mathbf{i}(z) e^{j \omega t} \right] = \text{Re} \left[ -\frac{j V_0 \cos \theta k z}{zj \sin \theta k} e^{j \omega t} \right] = -\frac{V_0 \cos \theta k z \text{cos} \omega t}{zj \sin \theta k}
\]

\[
\mathbf{i}(z,t) = \text{Re} \left[ \frac{\mathbf{i}(z)}{zj \sin \theta k} e^{j \omega t} \right] = \text{Re} \left[ -\frac{j V_0 \cos \theta k z}{zj \sin \theta k} e^{j \omega t} \right] = \frac{V_0 \cos \theta k z \text{cos} \omega t}{zj \sin \theta k}
\]

Resonance: \( \sin \theta k = 0 \Rightarrow \theta k = n \pi \Rightarrow \omega = n \theta k \), \( n = 1, 2, 3, \ldots \)

Complex impedance: \( Z(z) = \frac{\mathbf{f}(z)}{\mathbf{i}(z)} = -\frac{V_0}{j \omega L} \)

\[
Z(z = -L) = \begin{cases} +j \frac{V_0}{\omega L}, & n = 1, 2, 3, \ldots \pi \angle & \text{short-circuit} \\ n = 1, 2, 3, \ldots \end{cases}
\]

\( n - \frac{1}{2} \pi < k < n \pi, \quad Z(z = -L) = -j \frac{V_0}{\omega L}, \quad X > 0 \) (positive reactance, inductive)

\[
\lim_{k \rightarrow 1} Z(z = -L) = jL \quad \text{inductive}
\]

\[
\lim_{k \rightarrow 1} V(z,t) = V_0 \omega \cos \omega t \Rightarrow V(z = -L,t) = V_0 \omega \cos \omega t = (L \omega) \frac{d}{dt} (\mathbf{i}(z = -L,t))
\]

\[
i(z,t) = \frac{V_0}{\omega L} \sin \omega t \quad i(z = -L,t) = \frac{V_0 \sin \omega t}{(L \omega) L} \quad \text{capacitive}
\]

\[
\lim_{k \rightarrow 1} V(z,t) = -j V_0 \text{cos} \omega t \Rightarrow V(z = -L,t) = V_0 \text{cos} \omega t = (L \omega) \frac{d}{dt} (\mathbf{i}(z = -L,t))
\]

\[
i(z,t) = \frac{V_0}{\omega L} \sin \omega t \quad i(z = -L,t) = \frac{V_0 \sin \omega t}{(L \omega) L} \quad \text{capacitive}
\]
C. Open Circuit Line \( (i(t=0,t)=0), v(z=l,t)=V_0 \sin \omega t \)
\[ i(z) = V_0 \left[ \hat{V}_+ e^{-j \omega z} - \hat{V}_- e^{j \omega z} \right] \Rightarrow i(z=0)=0 = V_0 \left[ \hat{V}_+ - \hat{V}_- \right] = \hat{V}_+ \]
\[ i(z=-l) = jV_0 = \hat{V}_+ e^{j \omega l} + \hat{V}_- e^{-j \omega l} = \hat{V}_+ (e^{j \omega l} + e^{-j \omega l}) = 2 \hat{V}_+ \text{coke} \]
\[ \hat{V}_+ = \dot{V}_- = \frac{jV_0}{2 \text{coke}} \]
\[ \hat{V}(z) = \frac{jV_0}{2 \text{coke}} (e^{-j \omega z} + e^{j \omega z}) = -jV_0 \frac{\text{coke} \hat{V}}{\text{coke}} = -jV_0 \text{coke} \]
\[ i(z) = -jV_0 V_0 \left( e^{-j \omega z} - e^{j \omega z} \right) = \frac{(-jV_0 V_0)(z j \sin \omega z}{2 \text{coke}} \]
\[ = -\frac{V_0 V_0 \sin \omega z}{\text{coke}} \]
\[ V(z,t) = \text{Re} \left[ \hat{V}(z) e^{j \omega t} \right] = V_0 \frac{\text{coke} \sin \omega t}{\text{coke}} \]
\[ i(z,t) = \text{Re} \left[ i(z) e^{j \omega t} \right] = -\frac{V_0 V_0 \sin \omega z}{\text{coke}} \text{coke} \]

Resonance: \( \text{coke} = 0 \Rightarrow \text{le} = (2n-1) \frac{\pi}{2}, n=1,2,3,... \)
\[ \omega_n = \frac{(2n-1) \pi}{2} \]

Complex Impedance
\[ Z(z) = \frac{\hat{V}(z)}{i(z)} = Z_0 \frac{\cos \omega z}{i(z)} \]
\[ Z(z=-l) = -j Z_0 \frac{\cos \omega l}{i(z)} \]

\[ \lim_{t \to 0} \frac{V(z,t)}{V_0 \text{smut}} \]
\[ \lim_{t \to 0} i(z,t) = -V_0 \frac{\text{le} \text{coke}}{\text{coke}} \]
\[ i(z=-l,t) = (C l) \frac{\text{coke} \text{smut} (z=-l,t)}{dt} \]
Open Circuit Line

\[ Z(s) = -\frac{V_0 \cos \alpha s}{\cos kl} \]

\[ I(s) = \frac{V_0}{V_o} \left( \frac{V_0 \cos kl}{\cos kl} \right) \]

\[ I(s) = -\omega V_0 s \]

Impedance for Short and Open Circuited Lines

Figure 8-15