1 General Comments

1. Importance of geometric interpretation
   This course is a rigorous introduction to the theory of linear optimization and its extensions. Hence, rigorous math will be central to the course. However, the geometric interpretations of the material are equally as important and often lend great insight into the problems. Part of the appeal of linear optimization is the elegance of the geometry involved. I urge you to develop your geometric intuition regarding these problems as they will help you throughout the course.

2. Strict vs. non-strict inequalities
   We always use non-strict inequalities. This has to do with openness and closedness of sets. Think of a set described by intersections of strict inequalities. So the set does not include its boundary. Hence, there would never be an optimal feasible solution, just a sequence of points which converges to a point which produces the optimal cost, but is not feasible. Ie, we would be looking at the inf, instead of the min.

3. Power of mathematical programming formulations
   Take a moment to appreciate the powerful machinery that we are introducing here. We define variables and constraints and objectives to model some real-life problem. And then somehow (using the simplex method, for example), we manage to solve the problem algorithmically. However, one need not know anything about the simplex method or other similar algorithms in order to use linear programming. If you can model and formulate problems correctly, then you have already gained a valuable tool.

2 Basic Notation

1. What are the relationships between \( A, x, b, c \)?
   **Answer:** Typically, we work with an \( m \times n \) matrix \( A \). Hence, \( x, c \in \mathbb{R}^n \), and \( b \in \mathbb{R}^m \). Typically, we assume \( m \leq n \) in standard form problems, since we assume the rows are linearly independent (why do we do this?).

2. What does \( Ax = b \) mean?
   **Answer:** Think of some vector \( b \in \mathbb{R}^m \). Hitting the matrix \( A \) with the vector \( x \) means we are taking linear combinations of the columns of \( A \) (as determined by the components of \( x \)) to try and make the vector \( b \). If no such \( x \) exists, that means \( b \) is outside the subspace spanned by the columns of \( A \). If \( x \geq 0 \), the we are looking at the cone generated by the columns of \( A \). We’ll see more of this in future lectures.
3 Linear Algebra review

3.1 Key Terms and Concepts

1. Linear independence
   \textbf{Answer:} A set of vectors \( x_1, \ldots, x_k \in \mathbb{R}^n \) are \textit{linearly independent} if the only solution to
   \[ \sum_{i=1}^{k} a_i x_i = 0 \]
   is \( a_1 = \cdots = a_k = 0 \). Note that in \( \mathbb{R}^n \), any set of linearly independent vectors must have cardinality at most \( n \).

2. Subspace
   \textbf{Answer:} A set \( S \) is a \textit{subspace} if for every \( x, y \in S \) and every \( \lambda, \mu \in \mathbb{R} \), we have \( \lambda x + \mu y \in S \). If \( S \neq \mathbb{R}^n \), we call \( S \) a proper subspace. Note that every subspace, by definition, contains the origin. In \( \mathbb{R}^3 \) think of planes, and in \( \mathbb{R}^2 \), think lines.

3. Span
   \textbf{Answer:} The \textit{span} of a set of vectors \( x_1, \ldots, x_k \) is the subspace formed by all linear combinations of the vectors, i.e.
   \[ \text{span}(x_1, \ldots, x_k) = \{ y \mid y = \sum_{i=1}^{k} a_i x_i, \ a_i \in \mathbb{R} \} \]
   Why is this a subspace?

4. Basis
   \textbf{Answer:} A \textit{basis} for a subspace \( S \) is a set of linearly independent vectors whose span is \( S \). Think of a basis as the smallest set necessary to generate a subspace.

5. Dimension
   \textbf{Answer:} The \textit{dimension} of a subspace is the number of vectors in its basis. Is it possible to have multiple bases of different cardinalities?

6. Matrix range
   \textbf{Answer:} The \textit{range} of a matrix, or \textit{column space}, is denoted by
   \[ \mathcal{R}(A) = \{ Ax \mid x \in \mathbb{R}^n \} \]
   This is simply the span of the columns of \( A \), and hence is a subspace.

7. Matrix nullspace
   \textbf{Answer:} The \textit{nullspace} of a matrix \( A \) is the set of all vectors orthogonal to the rows of \( A \):
   \[ \mathcal{N}(A) = \{ x \mid Ax = 0 \} \]
   Clearly \( \mathcal{N}(A) \) is also a subspace.

8. Matrix rank
   \textbf{Answer:} There are many equivalent definitions of the \textit{rank} of a matrix. Perhaps the most fundamental is
   \[ \text{rank}(A) = \text{dim}(\mathcal{R}(A)) \].

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This is equivalent to the maximum number of independent columns (or rows) of the matrix \( A \). This implies that \( \text{rank}(A) \leq \min(m, n) \). A matrix is said to have full-rank when this bound is tight. Note also that, for \( B \in \mathbb{R}^{m \times r} \), \( C \in \mathbb{R}^{r \times n} \), we have

\[
\text{rank}(BC) \leq \min(\text{rank}(B), \text{rank}(C)) \leq r.
\]

Finally, a handy “conservation of dimension” equation is

\[
\text{rank}(A) + \dim(N(A)) = n,
\]

for \( A \in \mathbb{R}^{m \times n} \).

9. Affine subspace

**Answer:** An affine subspace is just a subspace shifted by a nonzero vector. Formally, an affine subspace \( S \) can be written

\[
S = \{ y \mid y = x + x_0, x \in S_0 \}
\]

where \( S_0 \) is a subspace. As an example, the set of solutions (provided one exists) to \( Ax = b \) is an affine subspace (why?).

### 3.2 Inversion and Transposition Properties

These are good facts to know:

1. \((AB)^T = B^T A^T\)
2. \((AB)^{-1} = B^{-1} A^{-1}\)
3. \((A^T)^{-1} = (A^{-1})^T\)

The third one follows from the first two (how?).

### 4 Examples

Give an explicit solution of each of the following LPs, describing the conditions under which the problem is unbounded, infeasible, etc.

1. Minimizing a linear function over a rectangle

\[
\begin{align*}
\min & \quad \mathbf{c}' \mathbf{x} \\
\text{st} & \quad \mathbf{l} \leq \mathbf{x} \leq \mathbf{u},
\end{align*}
\]

where \( \mathbf{l} \leq \mathbf{u} \).

**Answer:** The condition \( \mathbf{l} \leq \mathbf{u} \) guarantees we always have a feasible solution (e.g. \( \mathbf{x} = \mathbf{l} \)). We need to look at \( \mathbf{c} \) componentwise. Clearly, if \( c_i < 0 \), then we want to make the corresponding component of \( x_i \) as large as possible, and vice versa. Thus, the optimal solution \( \mathbf{x}^* \) is given by

\[
x^*_i = \begin{cases} 
\mathbf{l}_i & \text{if } c_i > 0, \\
\mathbf{u}_i & \text{if } c_i \leq 0.
\end{cases}
\]

The optimal value is given by

\[
\mathbf{c}' \mathbf{x} = \sum_{\{ i | c_i > 0 \}} c_i \mathbf{l}_i + \sum_{\{ i | c_i \leq 0 \}} c_i \mathbf{u}_i.
\]
2. Minimizing a linear function over the standard simplex

\[
\begin{align*}
\min & \quad c'x \\
\text{st} & \quad e'x = 1, \\
& \quad x \geq 0
\end{align*}
\]

where \( e \) is the vector of all 1’s

**Answer:** This problem is also always feasible (e.g. \( x_i = 1/n \), for all \( i \)). The feasible set has \( n \) vertices (what are they?), and for a given value of \( c \), at least one of them will be an optimal solution. Let \( j = \arg \min_{i=1,\ldots,n} (c_i) \). Then an optimal solution \( x^* \) is simply

\[
x^*_i = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

The optimal value is just \( c_j \). (When are there multiple optimal solutions?).

3. Minimizing a linear function over a halfspace

\[
\begin{align*}
\min & \quad c'x \\
\text{st} & \quad a'x \leq b,
\end{align*}
\]

where \( a \neq 0 \).

**Answer:** This problem is always feasible since there is only one inequality constraint. We distinguish between two cases (for both, simple drawings in \( \mathbb{R}^2 \) are helpful visual aids):

(a) \( -c \parallel a \). In this case, the set of optimal solutions lies along the affine subspace \( a'x = b \) (since we wish to travel as far as possible in the \( -c \) direction, but \( -c \parallel a \), so we can go no further than the boundary of the feasible set). The optimal value is precisely \( b ||c||/||a|| \) in this case.

(b) \( -c \not\parallel a \). Here, from any feasible point, we can find a component in the \( -c \) direction and travel infinitely far. Thus, the optimal value is unbounded below.

4. Minimizing a linear function over an affine set

\[
\begin{align*}
\min & \quad c'x \\
\text{st} & \quad Ax = b
\end{align*}
\]

**Answer:** Clearly the problem is feasible if and only if the system \( Ax = b \) has a solution. That is, if \( b \in \mathcal{R}(A) \). Assuming this is so, then recall that the set of feasible points will be an affine subspace. In particular, if \( x_0 \) is any feasible point, then we can write the entire feasible set \( P \) as

\[
P = \{ x_0 + z \mid z \in \mathcal{N}(A) \}.
\]

Since the nullspace is a subspace, we can go arbitrarily far in any direction in the nullspace. Thus, if we can find some \( z \in \mathcal{N}(A) \) such that \( c'z \neq 0 \), then the problem will be unbounded (either by heading in the \( z \) or \( -z \) direction). So the condition for a bounded optimal value is \( c \perp \mathcal{N}(A) \), and the optimal value in such a case will be \( c'x_0 \), where \( x_0 \) is any solution to \( Ax = b \).

5. Acknowledgements

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