1. (24 points) For each one of the statements below, state whether it is true or false. Include a few supporting sentences or a drawing, enough to convince us that you are not guessing the answer, but not a comprehensive, rigorous formal justification.

(Correct answer : +3 points, wrong answer : −1 points, no answer : 0 points.)

(a) The union of two polyhedra is a polyhedron.

(b) Every polyhedron \( P \) can be written in standard form \( P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \) and can be written in geometric form \( P = \{ x \in \mathbb{R}^n : Ax \geq b \} \).

(c) If there exists a vector \( q \neq 0 \) for which \( Aq = 0 \), then the polyhedron \( \{ x : Ax \geq b \} \) doesn’t have any vertex.

(d) Suppose that the polyhedron \( \{ x : Ax \geq b \} \) is non-empty and bounded, then \( x = 0 \) is the only vector for which \( Ax = 0 \).

(e) The set of optimal solutions to a linear optimization problem is a polyhedron.

(f) At an optimal solution of a linear optimization problem in \( \mathbb{R}^n \) there are at least \( n \) active constraints.

(g) Let \( x \) be a basic feasible solution associated with some basic matrix. If \( x \) is the unique optimal solution, then the reduced cost of every nonbasic variable is positive.

(h) If a linear optimization problem is feasible, then so is its dual and the optimal costs are equal.

2. (24 points) We are given a matrix \( A \) and a vector \( b \). The systems of equations that we would like to solve, \( Ax = b \), doesn’t have a solution. We therefore turn our attention to the problem of minimizing \( \|Ax - b\|_1 \) where the norm \( \| \cdot \|_1 \) is defined by \( \|x\|_1 = \sum_{i=1}^{n} |x_i| \).

(a) Write this optimization problem as a linear optimization problem.

(b) Derive a lower bound on the number of residues \( r_i = a'_i x - b_i \) that are equal to zero at an optimal vertex.

(c) We now want to solve the same problem but with the additional constraint that \( \|x\|_\infty \leq 1 \) (the norm \( \| \cdot \|_\infty \) is defined by \( \|x\|_\infty = \max_{i=1}^{n} |x_i| \)). Can that problem be formulated as a linear optimization problem? Either do so or explain why it cannot be done.
3. (28 points) You want to maximize your profit by optimally allocating your resources. The consultant A you hire for doing this formulates your problem as a linear optimization problem, and so does a second consultant B. The models of A and B use the same solution space \( \mathbb{R}^n \) but have different constraints and objectives. In particular, they lead to different sets of optimal solutions. In order to maximize your chances of success you would like to choose an allocation that is optimal for both models.

(a) You have access to a computer that has limited memory and that can only solve linear optimization problems with \( n \) variables. How would you proceed for checking that an optimal allocation (for both A and B) is possible by calling this procedure a few times?

(b) Due to a system crash, the linear optimization procedure doesn’t work anymore but you now have access to a procedure that checks whether or not a given polyhedron is empty; this procedure works for problems of arbitrary dimensions. Can you solve your optimization problem with just one call of that procedure?

4. (24 points) An alternative to the phase-I method for solving the LP

\[
(P) \quad \text{minimize} \quad c'x \\
\text{subject to} \quad Ax \leq b
\]

is the big-M-method, in which we solve the auxiliary problem

\[
(A) \quad \text{minimize} \quad c'x + Mt \\
\text{subject to} \quad Ax \leq b + t1 \\
t \geq 0
\]

In this auxiliary problem \( M > 0 \) is a parameter and \( t \) is an auxiliary variable.

(a) Find a feasible solution for the auxiliary problem \((A)\).

(b) Derive the dual of \((A)\).

(c) Prove the following property. If \( M > 1'p^* \), where \( p^* \) is an optimal solution of the dual of \((A)\), then the optimal \( t \) in \((A)\) is zero, and therefore the optimal \( x \) in \((A)\) is also an optimal solution of \((P)\).