Do all of the following problems. Due in recitation on Wednesday, 19 September 2007.

Warmups

1. Finding unit sample (impulse) responses
   Determine the unit sample (impulse) responses of the systems represented by the following system functionals.
   a. \((1 - \mathcal{R})^{-2}\)

   This system can be written as the cascade of two accumulators:
   \[
   \frac{Y}{X} = \frac{1}{(1 - \mathcal{R})^2} = \left( \frac{1}{1 - \mathcal{R}} \right) \left( \frac{1}{1 - \mathcal{R}} \right).
   \]
   In response to a unit impulse, the first accumulator will generate a unit step function, i.e., the output will be one for all time indices \(n \geq 0\). In response to the unit step, the second accumulator will count upwards from 1 so the output \(y[n]\) will be \(n + 1\) for \(n \geq 0\).

   b. \((1 + \mathcal{R})^3/8\)

   \[
   \frac{Y}{X} = \frac{(1 + \mathcal{R})^3}{8} = \frac{(1 + 2\mathcal{R} + \mathcal{R}^2)(1 + \mathcal{R})}{8} = \frac{1 + 3\mathcal{R} + 3\mathcal{R}^2 + \mathcal{R}^3}{8}.
   \]
   The output will be the sequence \(1/8, 3/8, 3/8, 1/8, 0, 0, 0, \ldots\).

2. Working with block diagrams
   Consider the system represented by the following block diagram.

   ![Block Diagram](image)

   a. Find a difference equation that characterizes this system.

   Let \(W\) represent the input of the delay element. Then \(W = X - 0.5\mathcal{R}W\) and \(Y = W + \mathcal{R}W\). The overall system functional is then
   \[
   \frac{Y}{X} = \frac{Y}{W} \frac{W}{X} = \frac{1 + \mathcal{R}}{1 + 0.5\mathcal{R}}.
   \]

   b. Determine the unit sample response of this system.
First compute

\[ W = \frac{1}{1 + 0.5R} X = \left( 1 - \frac{1}{2} R + \frac{1}{4} R^2 - \frac{1}{8} R^3 + \frac{1}{16} R^4 - \cdots \right) X. \]

Since \( X \) is the unit sample, \( W \) is the sequence \( 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots \). Then \( Y = (1 + R)W \) so we add \( W \) to a right-shifted version of \( W \):

\[ W : \quad 1 \quad -\frac{1}{2} \quad \frac{1}{4} \quad -\frac{1}{8} \quad \frac{1}{16} \quad \ldots \]
\[ RW : \quad 0 \quad 1 \quad -\frac{1}{2} \quad \frac{1}{4} \quad -\frac{1}{8} \quad \ldots \]
\[ Y : \quad 1 \quad \frac{1}{2} \quad -\frac{1}{4} \quad \frac{1}{8} \quad -\frac{1}{16} \quad \ldots \]

3. Finding a system

a. Determine a system whose output is \( 10, 1, 1, 1, 1, \ldots \) when the input is \( 1, 1, 1, 1, 1, \ldots \). Determine the difference equation and block diagram representations for this system.

Notice that \( Y = 10X - 9RX \). This relation suggests the following difference equation

\[ y[n] = 10x[n] - 9x[n - 1] \]

and block diagram

![Block diagram for part a](image)

b. Determine a system whose output is \( 1, 1, 1, 1, 1, \ldots \) when the input is \( 10, 1, 1, 1, 1, \ldots \). Determine the difference equation and block diagram representations for this system.

The difference equation for the inverse relation can be obtained by interchanging \( y \) and \( x \) in the previous difference equation to get

\[ x[n] = 10y[n] - 9y[n - 1]. \]

So

\[ y[n] = \frac{9y[n - 1] + x[n]}{10}, \]

which has this block diagram

![Block diagram for part b](image)

c. Compare the difference equations in parts a and b. Compare the block diagrams in parts a and b.
The difference equations for parts a and b have exactly the same structure. The only difference is that the roles of \( x \) and \( y \) are reversed. The block diagrams have similar parts, but the topologies are completely different. The first is acyclic and the second is cyclic.

4. Finding modes
The following block diagram describes the relation between two discrete-time signals: \( x[n] \) and \( y[n] \).

![Block Diagram]

**a.** How many different modes does this system have?

Let \( W \) represent the output of the leftmost adder so that

\[
W = X + RY.
\]

The output of the middle adder is then \((1 - R)W\). The output of the rightmost adder is

\[
Y = \frac{7}{2}(1 - R)W + R^2W.
\]

Substituting the former equation for \( W \) into the latter yields

\[
Y = (\frac{7}{2} - \frac{7}{2}R + R^2)(X + RY).
\]

Solving for the functional, we find

\[
\frac{Y}{X} = \frac{\frac{7}{2} - \frac{7}{2}R + R^2}{1 - \frac{1}{2}R + \frac{7}{2}R^2 - R^3}.
\]

The denominator of this functional can be factored to give

\[
\frac{Y}{X} = \frac{\frac{7}{2} - \frac{7}{2}R + R^2}{(1 - \frac{1}{2}R)(1 - R)(1 - 2R)}.
\]

The system has three modes, one corresponding to each factor in the denominator.

**b.** Determine closed-form expressions for each mode. [Do not try to determine the amplitudes of the modes. The amplitudes depend on the input \( X \), which is not specified.]

The roots of the denominator (above) are \( \frac{1}{2}, 1, \) and \( 2 \). Therefore the three modes have the forms \( \frac{1}{2^n}, 1^n, \) and \( 2^n \) for \( n \geq 0 \).
Problems

5. Drug dosing

When you start a medicine, the doctor often says ‘Take double the usual dosage for the first dose, then follow the schedule.’ In this problem you study this advice to take a so-called loading dose.

A simple model of the body is a tank of blood from which drug exits at a rate proportional to its concentration and into which drug doses instantly arrive. Assume that the patient takes the drug once every 8 hours, and that the concentration of drug in the blood is measured shortly after taking each dose. Draw the block diagram and give the system functional of a discrete-time system with this behavior. Choose the parameter(s) of the system so that, in the absence of new drug, two-thirds of the drug is filtered out (say, by the kidneys) every 8 hours.

Ideally, the level of drug in the blood would instantly go to the desired level and remain steady. Assume that you take the medicine every 8 hours. What dosage schedule will produce the ideal behavior? Interpret your answer in terms of loading doses, whether for or against the idea.

One-third of the drug remains in the bloodstream after 8 hours (one dosing interval). Therefore

\[ y[n] = \frac{1}{3} y[n-1] + x[n] \]

and the corresponding functional is

\[ \frac{Y}{X} = \frac{1}{1 - \frac{1}{3} R} \]

The block diagram is

![Block Diagram]

We want an input \( x[n] \) that produces an output \( y[n] = 1 \) for \( n \geq 0 \). Notice that we can just run the difference equation backwards using the desired output:

\[
\begin{align*}
x[n] &= y[n] - \frac{1}{3} y[n-1] \\
x[0] &= y[0] - \frac{1}{3} y[-1] = 1 - 0 = 1 \quad \text{(no drug in body at } n = -1) \\
x[1] &= y[1] - \frac{1}{3} y[0] = 1 - \frac{1}{3} = \frac{2}{3} \\
x[2] &= y[2] - \frac{1}{3} y[1] = 1 - \frac{1}{3} = \frac{2}{3} \\
x[3] &= y[3] - \frac{1}{3} y[2] = 1 - \frac{1}{3} = \frac{2}{3} \\
\cdots
\end{align*}
\]

Thus the desired dosing is to take one unit initially and \( \frac{2}{3} \) of a unit for all subsequent doses.
This schedule is an example of using a loading dose. The first dose is somewhat higher – here 1.5 times higher – than the steady dosage, in order to jump-start the drug's level to its steady-state value.

This discrete-time analysis ignores the behavior between sampling intervals, i.e. between the 8-hour dosage intervals. We will return to this problem when we study continuous-time systems to see whether or how much the drug's level in the blood overshoots the ideal of a step function.

6. Experimental mathematics to debug a black box

You would like to characterize a system but all you know is its unit-sample (impulse) response, which is available as \( n, y[n] \) pairs in the Appendix, and is available electronically on our course website as `black_box.tsv`. Use peeling away and educated guessing (see the R04 notes) to find the modes and their amplitudes, and therefore find a closed form for the output signal.

What is the system functional and the corresponding difference equation? [Note: Feel free to use a computer, graphing calculator, or paper and pencil.]

The following Python code reads the values from “black_box.tsv” and computes successive ratios \( y[n+1]/y[n] \).

```python
from scipy import *

y = zeros(100)
for line in open("black_box.tsv","r"):
    (n,fn) = line.split()
    y[int(n)] = float(fn)

y = y[0:int(n)+1] # chop of unused part of y
print y[1:] / y[:-1] # y[n+1]/y[n]
```

The ratios converge to 3 so subtract out \( 3^n \). Continue the preceding Python code by adding these lines:

```python
N = arange(len(y)) # N = [0, 1, 2, ... ,max_n]
print y - 3.0**N # try taking out 3^n
```

The numbers still grow quickly. We could be off by a multiplicative constant, since the \( 3^n \) mode could have any amplitude. To find the amplitude, divide out \( 3^n \):

```python
print y / 3.0**N # try dividing out 3^n
```

The result is 3. So subtract out \( 3 \cdot 3^n = 3^{n+1} \) and compute the successive ratios of that signal:

```python
z = y - 3.0**(N+1) # take out first mode
print z
print z[1:] / z[:-1] # z[n+1]/z[n]
```

Now we get a ratio of 2. The first few samples of \( Z \) are \(-2, -4, \) and \(-8\). So \( z[n] = -2^{n+1} \), which means that \( Y \) is given by

\[
y[n] = 3^{n+1} - 2^{n+1}
\]

The combination of partial fractions that generates these modes is

\[
\frac{3}{1-3R} - \frac{2}{1-2R}
\]
Adding the fractions gives this form of the system functional

\[ \frac{Y}{X} = \frac{1}{1 - 5R + 6R^2}. \]

The corresponding difference equation is

\[ y[n] = 5y[n-1] - 6y[n-2] + x[n]. \]

Let’s check whether this difference equation generates the input data when \( X \) is the unit sample (impulse). Indeed, it gives the output signal 1, 5, 19, 65, ….

7. Making sherry: redux

Reconsider the three-barrel system for “Making Sherry” in Homework #1 (Problem 5).

a. Express the difference equations that describe the three-barrel solera system using system functionals. Sketch a block diagram for the three-barrel solera system.

The difference equations for the three-barrel sherry system can be expressed in operator notion as

\[ Y = 0.5RC \]

\[ (1 - 0.5R)C = 0.5RB \]

\[ (1 - 0.5R)B = 0.5RA \]

\[ (1 - 0.5R)A = X \]

Therefore

\[ \frac{Y}{C} = 0.5R \]

\[ \frac{C}{B} = \frac{0.5R}{1 - 0.5R} \]

\[ \frac{B}{A} = \frac{0.5R}{1 - 0.5R} \]

\[ \frac{A}{X} = \frac{1}{1 - 0.5R} \]

These system functionals correspond to the following block diagram.

b. Rewrite the block diagram so that the main body is the cascade of three identical systems. Determine the system functional for this block diagram.
The system in part a is a cascade of three of the following systems.

\[ Y = \left( 0.5R \frac{1 - 0.5R}{1 - 0.5R} \right)^3. \]

c. Determine the system functional for a ten-barrel solera system analogous to the system functional in part b.

\[ Y = \left( 0.5R \frac{1 - 0.5R}{1 - 0.5R} \right)^{10}. \]

d. Assume that one unit of tracer is added to the freshly crushed grapes during year 0 and no tracer is added in any other year. Write a program to determine how much tracer is in the sherry bottled in year \( n \) for a \( B \)-barrel solera system, where \( B \) is a parameter to your program.

```python
# compute impulse response of a many-barrel sherry-making system
from scipy import *
import pylab as p
from sys import argv

# use 20 barrels, unless overridden on the cmd line
try : n = int(argv[1])
except : n = 20

# signals[k] is the input to barrel k+1. In particular, signals[0] is # the input signal. signals[-1], which is the last element of the # signals vector, is the output signal, which is the amount of tracer # bottled. At every time step the signals vector is updated.
signals = zeros(n+1) # +1 to include the input signal
signals[0] = 1 # 1 unit of tracer is input at time 0
bottledtracer = [] # for storing output signal vs time

# indicies to slice the signals vector. It's 0,0,1,2,3, ..., n-1.
indicies = [0] + range(n)

while len(bottledtracer) < 3*n: # reaches a max before 3n years
    bottledtracer.append(signals[-1]) # save output sample for plotting
    signals = (signals + signals[indicies])/2.0
    signals[0] = 0 # no more tracer after year 0
```
Generally, the amount of tracer that is bottled in year $n$ first increases with $n$ and then decreases. Demonstrate this trend by plotting results for a 20-barrel solera.

The output is zero for many years (why?) and is a maximum in years 38 and 39.

**Hours**

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.
Appendix

Data for problem 5 (also available as a .tsv file on the course website).

0  1
1  5
2 19
3  65
4 211
5  665
6 2059
7  6305
8 19171
9  58025
10 175099
11 527345
12 1586131
13  4766585
14 14316139
15  42981185
16 129009091
17 387158345
18 1161737179
19 3485735825
20 10458256051
21  31376865305
22  94134790219
23 282412759265
24  847255055011
25 2541798719465
26  7625463267259
27 22876524019505
28  68629840493971
29 205890058352825
30  617671248800299
31  1853015893884545
32  5559051976620931
33  16677164519797385
34 50031510739261339