Multiple Representations of Discrete-Time Systems

Now you know four representations of discrete-time systems.

**Verbal descriptions:** preserve the underlying physics.

“To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.”

**Difference equations:** mathematically compact.

\[ y[n] = x[n] - x[n-1] \]

**Block diagrams:** illustrate signal flow paths.

\[ \begin{align*}
    &x[n] \\
    \downarrow &-1 \\
    \downarrow &\text{Delay} \\
    &y[n] \\
\end{align*} \]

**Operator representations:** analyze systems as polynomials.

\[ Y = (1 - R) X \]

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Last Time

The persistent response of a complicated system to a transient input signal can be decomposed into a sum of simpler parts called **modes**.

\[ \frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots} \]

Factor denominator:

\[ \frac{Y}{X} = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R)\cdots} \]

Partial fractions:

\[ \frac{Y}{X} = \frac{C_0}{1 - p_0 R} + \frac{C_1}{1 - p_1 R} + \frac{C_2}{1 - p_2 R} + \cdots + D_0 + D_1 R + D_2 R^2 + \cdots \]
Modal Decomposition

The sum corresponds to parallel paths through a block diagram representation of the system.

\[
\begin{align*}
X & \rightarrow Y_1 \rightarrow Y \\
X & \rightarrow Y_2 \rightarrow Y
\end{align*}
\]

Today: Poles and Zeros

Cascade (series) decomposition: poles and zeros.

Factoring the numerator and denominator of the system functional breaks the system into parts that are multiplied together.

\[
\begin{align*}
\frac{Y}{X} & = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots} \\
\frac{Y}{X} & = G \frac{(1 - z_0 R)(1 - z_1 R)(1 - z_2 R)(1 - z_3 R) \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots}
\end{align*}
\]

Poles and Zeros

We can think of the factors as subsystems that are connected in series.

\[
\begin{align*}
\frac{Y}{X} & = \frac{b_0 + b_1 R + b_2 R^2 + b_3 R^3 + \cdots}{1 + a_1 R + a_2 R^2 + a_3 R^3 + \cdots} \\
\frac{Y}{X} & = G \frac{(1 - z_0 R)(1 - z_1 R)(1 - z_2 R)(1 - z_3 R) \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots}
\end{align*}
\]

cascade subsystems ↔ multiply subsystem functionals
**Poles and Zeros**

Divide and conquer: figure out how each of the subsystems work, then cascade the subsystems to describe how the system works.

\[
X \rightarrow G \rightarrow \frac{1}{1-p_0R} \rightarrow \frac{1}{1-p_1R} \rightarrow (1-z_0R) \rightarrow (1-z_1R) \rightarrow \cdots \rightarrow Y
\]

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**Poles and Zeros**

The gain $G$ is easy. It produces an output signal that is $G$ times the input signal.

\[
Y = GX
\]

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**Poles and Zeros: Denominator Sections**

Each denominator factor contributes a cyclic signal flow path. Therefore, transient inputs can give rise to a persistent response called a mode.

\[
\frac{1}{1-p_0R}
\]

\[
x[n] = \delta[n]
\]

\[
y[n]
\]

The unit sample response is a geometric sequence.
Poles and Zeros: Numerator Sections

Each numerator factor contributes an acyclic signal path. Therefore, the response to a transient signal is transient.

\[ Y = (1 - z_0R)X \]

\( x[n] = \delta[n] \)

\( y[n] = \delta[n] - \frac{1}{2}\delta[n-1] \)

Poles and Zeros: Numerator Sections

Numerator and denominator sections have similar functional descriptions, with the roles of input and output exchanged.

\[ (1 - p_0R)Y = X \]

\[ Y = (1 - z_0R)X \]

If the unit sample response of a denominator section is a geometric sequence, then the response of a numerator section to a geometric sequence should be a unit sample!

Poles and Zeros: Numerator Sections

Calculate the response of a numerator section to a geometric sequence with base \( z_0 \).

\[ Y = (1 - z_0R)X \]

\( X: 1 \quad z_0 \quad z_0^2 \quad z_0^3 \quad ... \)

\( z_0RX: 0 \quad z_0 \quad z_0^2 \quad z_0^3 \quad ... \)

\( Y = X - z_0RX: 1 \quad 0 \quad 0 \quad 0 \quad ... \)
Poles and Zeros: Numerator Sections

The numerator section has a transient response to a persistent input! It “eats” geometric sequences with base $z_0$.

\[ Y = (1 - z_0 R) X \]

Poles and Zeros: Check Yourself

Design a system that turns the input signal $X: 1, 2, 3, 4, \ldots$ into output signal $Y: 1, 0, 0, 0, \ldots$

Poles and Zeros

Each of the boxes in the cascaded representation can be represented by a single number.

\[ Y = \frac{(1 - z_0 R)(1 - z_1 R)(1 - z_2 R)(1 - z_3 R) \cdots}{(1 - p_0 R)(1 - p_1 R)(1 - p_2 R)(1 - p_3 R) \cdots} X \]

The numbers are:
- gain: $G$
- zeros: roots of the numerator
- poles: roots of the denominator
**Poles and Zeros**

Characterize a system with a handful of numbers.

Example: Fibonacci System

\[ y[n] = y[n - 1] + y[n - 2] + x[n] \]

Factor the system functional,

\[ \frac{Y}{X} = \frac{1}{1 - \phi R - R^2} = \frac{1}{(1 - \phi R)(1 + \frac{1}{\phi} R)} \]

where \( \phi = (1 + \sqrt{5})/2 \).

\[ G = 1 \]

\[ p_0 = \phi \quad p_1 = -\frac{1}{\phi} \]

**Poles and Zeros**

Example 2

Factor system functional.

\[ \frac{Y}{X} = \frac{1 + R}{1 - R^2} = \frac{1 - z_0 R}{1 + R} \]

\[ \frac{1 - z_0 R}{1 + R} \left( 1 + \left( \frac{1}{2} + j \frac{\sqrt{3}}{2} \right) R \right) \left( 1 + \left( \frac{1}{2} - j \frac{\sqrt{3}}{2} \right) R \right) \]

\[ G = 1 \quad z_0 = -1 \]

\[ p_0 = 1 \quad p_1 = -\frac{1}{2} - j \frac{\sqrt{3}}{2} \quad p_2 = -\frac{1}{2} + j \frac{\sqrt{3}}{2} \]

**Pole-Zero Diagrams**

Represent the poles and zeros on the z-plane to graphically depict behaviors of modes.

Example: Fibonacci System

\[ p_0 = \phi \approx 1.618 \quad p_1 = -\frac{1}{\phi} \approx -0.618 \]
**Pole-Zero Diagrams**

Poles inside/outside the unit circle correspond to modes with amplitudes that decrease/increase with time.

Example: Fibonacci System

\[ p_0 = \phi \approx 1.618 \quad p_1 = -\frac{1}{\phi} \approx -0.618 \]

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**Example 2:**

\[ u[n] = \begin{cases} \frac{1}{2} + \frac{\sqrt{3}}{2} \end{cases} \]

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**Pole-Zero Diagrams**

Poles on/off the positive real axis correspond to modes that are monotonic/oscillatory in time.
Pole-Zero Diagrams

Poles on/off the positive real axis correspond to modes that are monotonic/oscillatory in time.

Example: Fibonacci System

\[ z_0 = \phi \approx 1.618 \]
\[ z_1 = \frac{-1}{\phi} \approx -0.618 \]

\[ \phi^n \]

\[ \text{Re} \ z \]
\[ \text{Im} \ z \]

\( n \)

\[ \phi^n \]

\( n \)

\[ -1 \ 0 \ 1 \ 2 \ 3 \ 4 \]

Poles on/off the positive real axis correspond to modes that are monotonic/oscillatory in time.

Example: Fibonacci System

\[ z_0 = \phi \approx 1.618 \]
\[ z_1 = \frac{-1}{\phi} \approx -0.618 \]

\[ \phi^n \]

\[ \text{Re} \ z \]
\[ \text{Im} \ z \]

\( n \)

\[ \phi^n \]

\( n \)

\[ -1 \ 0 \ 1 \ 2 \ 3 \ 4 \]

Poles and Zeros: Check Yourself

Determine the following numbers:

1. number of complex-valued modes in this system
2. number of oscillatory modes in this system
3. number of growing modes
4. number of decaying modes

Multiple Representations of Discrete-Time Systems

Now you know five representations of discrete-time systems.

**Verbal descriptions:** preserve the underlying physics.

“... record the first number, and then record successive differences.”

**Difference equations:** mathematically compact.

\[ y[n] = x[n] - x[n-1] \]

**Block diagrams:** illustrate signal flow paths.

**Operator representations:** analyze systems as polynomials.

\[ Y = (1 - \mathcal{R}) X \]

**Pole-Zero diagrams:** show factors of the system functional.
Just for fun! (It’s a lot easier than it looks.)

What’s the unit-sample response for this system:

$$\frac{Y}{X} = \left(\frac{1}{1-R^3}\right)^2$$