Last time

Feedback is useful for controlling systems.
Feedback → cyclic signal paths → persistent outputs (modes).
Pole-zero plots help visualize how modes depend on amount of feedback.

Feedback and Control: Robotic Driver

Design a system to automatically steer a car.

A moving car, velocity $V$

$p$

Assume a sensor reports position $p$ within the lane:

$p = 0$: in center
$p > 0$: right of center
$p < 0$: left of center
Use feedback to position the car.
Let $X$ represent the desired position in the lane (normally 0).
Turn the steering wheel ($\phi$) in proportion to the difference between the desired and current positions.

\[
\begin{align*}
X & \rightarrow \phi \rightarrow \text{steering system} \rightarrow P \\
& \downarrow \phi \quad -1
\end{align*}
\]

What’s the continuous-time relation between $\phi$ and $P$?
What’s the corresponding difference equation?

---

**Robotic Driver: Check Yourself**

Find the continuous-time relation between $\phi$ and $\theta$.

1. $\theta \propto \phi$
2. $\dot{\theta} \propto \phi$
3. $\theta \propto \dot{\phi}$
4. $\phi \propto \sin \theta$

---

Let $\phi$ represent the angle of the steering wheel. If you hold $\phi$ constant, you will drive in circles. The angle of the car, $\theta$, increases with time at a constant rate, proportional to $\phi$.

\[
\begin{align*}
\theta[n] &= \theta[n-1] + \beta \phi[n-1] \\
\Theta &= R\Theta + \beta R\Phi \\
\Theta &= \beta R \\
\Phi &= \frac{\beta R}{1 - R}
\end{align*}
\]
If the angle of the car ($\theta$) is not zero, then the position of the car within the lane ($p$) increases linearly with time.

$$p[n] = p[n - 1] + \gamma' V \sin \theta[n - 1]$$

$$\approx p[n - 1] + \gamma' V \theta[n - 1]$$

$$P = \hat{R}P + \gamma R \Theta$$

$$\frac{p}{\Theta} = \frac{\gamma R}{1 - \hat{R}}$$

Feedback and Control: Robotic Driver

To find the modes, put denominator of system functional in the form $(1 - p_0 \hat{R})(1 - p_1 \hat{R})$; then the modes are at $z = p_0$ and $z = p_1$.

$$\frac{Y}{X} = \frac{a \beta \gamma \hat{R}^2}{1 - 2\hat{R} + (1 + a \beta \gamma) \hat{R}^2}$$

where $K \equiv a \beta \gamma$. 

Solving:

$$\frac{K \hat{R}^2}{1 - 2\hat{R} + (1 + K) \hat{R}^2} = \frac{K \hat{R}^2}{(1 - p_0 \hat{R})(1 - p_1 \hat{R})}$$

$$p_0, p_1 = 1 \pm j \sqrt{K}$$
If $K = 0$, there is a double pole at $z = 1$.

If $K = 1$, there are complex poles at $z = 1 \pm j$. The output oscillates and diverges – the system is "unstable."

No values of $K = \alpha\beta\gamma$ result in acceptable performance.

Need a better controller.

The system is the cascade of two accumulators.
Feedback and Control: Robotic Driver

Try a difference controller (“derivative control”).

\[ Y = P = \frac{\gamma R}{1 - R} \Theta = \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} \Phi = a(1 - R) \frac{\beta R}{1 - R} \frac{\gamma R}{1 - R} E = \frac{a \beta \gamma}{1 - R} (X - Y) \]

Solving:

\[ \frac{Y}{X} = \frac{K R^2}{1 - R + K R^2} \quad \text{where} \quad K = a \beta \gamma \]

Feedback and Control: Robotic Driver

This is the same functional that we found for the “guiding person” problem (last lecture). The modes of the closed-loop system are stable for \( 0 < K < 1 \).
Feedback and Control: Robotic Driver

But can we really cancel the pole at $z = 1$ with a zero at $z = 1$?

$Y = P = \frac{\gamma R}{1-R}\Theta = \frac{\beta R}{1-R} \frac{\gamma R}{1-R}\Phi = \frac{\alpha(1-R)}{1-R}$

$E = \frac{\alpha\beta\gamma R^2}{1-R} (X-Y)$

Solving:

$\frac{Y}{X} = \frac{KR^2}{1-R + KR^2}$ where $K = \alpha\beta\gamma$

Feedback and Control: Robotic Driver

Try step-by-step analysis of the block diagram.

Feedback and Control: Robotic Driver

Try step-by-step analysis of the block diagram.

Feedback and Control: Robotic Driver

Try step-by-step analysis of the block diagram.

$p[0] = 0.5$  \hspace{1cm} \alpha = 0.5 \hspace{1cm} \beta = \gamma = 1

\theta[0] = 0.1

With time, the value of $\theta$ goes to zero, but the value of $p$ does not.
The angle $\theta$ goes to zero, but the position $p$ does not. This suggests that the closed loop response has a pole at $z = 1$ and a corresponding mode $1^n$, $n \geq 0$.

But this is not consistent with our analysis.

If $K = 0.5$, the poles are at $z = \frac{1}{2} \pm j\frac{1}{2}$. These poles correspond to decaying modes.
Feedback and Control: Robotic Driver

Is the persistent value of $p$ caused by pole-zero cancellation?

$\frac{\alpha(1-R)}{1-R}$

Analyse of the block diagram that results when we remove the parts that correspond to the pole and cancelling zero (red).

Now the value of $p$ also goes to zero with time.

Cancelling a Pole with a Zero

We did a step-by-step analysis of block diagrams for systems with the following open-loop poles and zeros:

- $z_0 = 1$ and $p_0, p_1 = 1$
- $p_0 = 1$

and we got different answers. This suggests that you cannot cancel a pole at 1 with a zero at 1.

What's going on?

Consider some simpler systems.
Cancelling a Pole with a Zero

Accumulate and difference.

\[ W = X + RW \]
\[ (1 - R)W = X \]
\[ Y = (1 - R)W \]
\[ Y = X \]

The pole and zero cancelled.

Cancelling a Pole with a Zero

Difference then accumulate.

\[ W = (1 - R)X \]
\[ Y = W + RY \]
\[ (1 - R)Y = W \]
\[ (1 - R)Y = (1 - R)X \]

The pole and zero almost cancelled.

Cancelling a Pole with a Zero

If the system is initially at rest, then \( Y = X \).

\[ (1 - R)X = (1 - R)Y \]
\[ y[0] = x[0] = 0 \]
\[ ... \]
Cancelling a Pole with a Zero

If the system is not initially at rest, then $Y$ and $X$ are not always equal.

\[
(1 - R)X = (1 - R)Y
\]

\[
y[0] = 5 \quad x[0] = 0
\]

\[
\]

\[
\]

\[
\ldots
\]

Feedback and Control: Robotic Driver

This system was not turned on at rest.

Check yourself: Why didn’t feedback make $p \rightarrow 0$?

\[
p[0] = 0.5 \quad \alpha = 0.5 \quad \beta = \gamma = 1
\]

\[
\theta[0] = 0.1
\]
Check yourself: Will feedback make $\theta \to 0$?

Feedback and Control: Robotic Driver

Derivative feedback never eliminates the constant outputs. Therefore, derivative controllers are not good for feedback systems that are intended to control position.

Need an even better controller.

Feedback and Control: Robotic Driver

Use combination of proportional and derivative control to eliminate unwanted persistent behavior.
Feedback and Control: Robotic Driver

Try step-by-step analysis of the block diagram.

Feedback and Control: Summary

Now you know about two kinds of controllers:

- proportional
- proportional plus derivative

Adding delays to a loop tends to destabilize the loop.
Adding accumulators to a loop tends to destabilize the loop.
Derivative feedback can help to stabilize such loops.