**Multiple Representations of Discrete-Time Systems**

**Verbal descriptions**: preserve the underlying physics.

“... record the first number, and then record successive differences.”

**Difference equations**: mathematically compact.

\[ y[n] = x[n] - x[n-1] \]

**Block diagrams**: illustrate signal flow paths.

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+-----------
|           |
|           |
|           |
|           | Delay
|           |
\[ x[n] \]
```

**Operator representations**: analyze systems as polynomials.

\[ Y = (1 - R) X \]

**Pole-Zero diagrams**: represent system functional in factored form.

**System functions**: represent systems as polynomials in \( z \).

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**Multiple Representations of CT Systems**

**Verbal descriptions** of continuous time systems typically specify relations among signals (e.g., inputs and outputs) and their rates of change.

“Water flows into a tank at rate \( r_0(t) \) and flows out at a rate that is proportional to the depth of water in the tank. Determine ...”

Verbal descriptions preserve the underlying physics of both DT and CT systems.
Multiple Representations of CT Systems

Differential equations are mathematically compact.

\[ y(t) - py(t) = x(t) \]

Notice that derivatives play a key role for CT systems, while time shifts played a key role for DT systems.

Block diagrams illustrate signal flow paths.

DT: block diagrams contained adders, scalers, delays, and advancers – sufficient for any system described by a linear difference equation with constant coefficients.

CT: block diagrams will contain adders, scalers, and integrators – sufficient for any system described by a linear differential equation with constant coefficients.

Why integrators and not differentiators?

One reason is –

In DT, delays (and advance elements) retain all of the “state” of a system (the minimum history (or future) that is required to iteratively solve for the next (or previous) time).

In CT, integrators serve the same function. The outputs of integrators hold the “state” in an analogous fashion.
Multiple Representations of CT Systems

We will define the \( A \) operator for functional analysis of CT systems.

Applying \( A \) to a CT signal generates a new signal that is equal to the integral of the first signal at all points in time.

\[ Y = AX \]

is equivalent to

\[ y(t) = \int_{-\infty}^{t} x(t) \, dt \]

for all time.

Check Yourself

Which of the following system functionals represents this system?

1. \( \frac{Y}{X} = \frac{A}{1 + pA} \)
2. \( \frac{Y}{X} = \frac{1}{p + A} \)
3. \( \frac{Y}{X} = \frac{A}{1 - pA} \)
4. \( \frac{Y}{X} = \frac{1}{p - A} \)

Multiple Representations of CT Systems

As with the \( R \) and \( L \) operators, \( A \) expressions can be manipulated as polynomials.

- Commutativity: \( A(1 - A)X = (1 - A)AX \)
- Distributivity: \( A(1 - A)X = (A - A^2)X \)
- Associativity: \( (1 - A)(2 - A)X = (1 - A)(A(2 - A))X \)
Multiple Representations of CT Systems

Does feedback produce persistent responses to transient inputs?

\[ y(t) = x(t) + py(t) \]

Find the impulse response.
First we have to define a CT impulse function.

\[ \delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t+\epsilon} \delta(\lambda) d\lambda \]

The CT impulse function can be thought of as a pulse of unit area whose width is reduced to zero.

\[ p_{1/2}(t) \]
\[ p_{1/4}(t) \]
\[ p_{1/8}(t) \]

\[ \delta(t) = 0 \quad \forall t \neq 0 \]

Its integral is one.

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \]

It follows that the integral of the unit impulse is the unit step function.

\[ u(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda = \begin{cases} 1; & t \geq 0 \\ 0; & \text{otherwise} \end{cases} \]
Check Yourself

Which of the following expressions gives the impulse response of this system?
\[ y(t) = x(t) + py(t) \]

1. \( e^{pt}, \quad t \geq 0 \)
2. \( e^{-pt}, \quad t \geq 0 \)
3. \( pe^{pt}, \quad t \geq 0 \)
4. \( pe^{-pt}, \quad t \geq 0 \)

Multiple Representations of CT Systems

Feedback introduces persistent response to transient inputs.

\[
\begin{align*}
\mathcal{X} &\quad \int_{-\infty}^{t} ( \cdot ) \, dt \quad \mathcal{Y} \\
\mathcal{F} &\quad x(t) \quad y(t)
\end{align*}
\]

Transient input: \( x(t) = \delta(t) \).

Persistent output: \( y(t) = e^{pt}, \quad t \geq 0 \).

If \( x(t) = \delta(t) \), then \( y(t) \) for \( t > 0 \) is a sum of
- \( \mathcal{A} \delta(t) = 1 \) (step function)
- \( p \mathcal{A} \delta(t) = pt \) (ramp function)
- \( p^2 \mathcal{A}^2 \delta(t) = (pt)^2/2 \)
- \( \cdots \)
- \( p^k \mathcal{A}^{k+1} \delta(t) = (pt)^k/k! \)
Multiple Representations of CT Systems

Determine persistent response directly from system functional.

\[
\frac{Y}{X} = \frac{A}{1-pA} = A \sum_{k=0}^{\infty} p^k A^k
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \sum_{k=0}^{\infty} p^k A^{k+1} \delta(t) = 1 + \cdots
\]

\[
y(t)
\]

\[
0 \quad t
\]
Check Yourself

Which of the following differential equations applies to the block diagram?

1. $\dot{y}(t) - 2y(t) = x(t) $
2. $\dot{y}(t) + 2y(t) = x(t) $
3. $\dot{y}(t) - 2y(t) = -x(t) $
4. $\dot{y}(t) - 2y(t) = -x(t) $

Check Yourself

Which of the following plots shows the impulse response of this differential equation?

$\dot{y}(t) + 2y(t) = x(t)$

Modes of Discrete-Time Systems
Multiple Representations of CT Systems

Consider the impulse responses that result for different values of $p$.

\[ x(t) \xrightarrow{\text{+}} \int_{-\infty}^{t} (\cdot) \, dt \xrightarrow{p} y(t) \]

\[ \frac{Y}{X} = \frac{A}{1 - pA} \]

Check Yourself

Indicate regions of the $p$-plane with convergent, divergent, monotonic, and non-monotonic growth.

Multiple Representations of CT Systems

Verbal descriptions: preserve the underlying physics.
“... the rate at which water flows from a leaky tank ...”

Differential equations: mathematically compact.
\[ \dot{y}(t) = x(t) + py(t) \]

Block diagrams: illustrate signal flow paths.

Operator representations: analyze systems as polynomials.
\[ (1 - pA)Y = AX \]