The goals of this chapter are:

- to turn a description of a first-order, continuous-time system into a differential equation;
- to understand the system’s behavior using a discrete-time approximation suitable for paper-and-pencil calculations; and
- to formalize this approximation into the general-purpose, forward-Euler method for numerically solving differential equations.

This chapter introduces a continuous-time system – the leaky tank – and its discrete-time approximation. Continuous- and discrete-time systems are often related because of how physics in general and computers in particular work. In physics, time is a continuous variable. Yet computers are discrete-time machines; more accurately, they are continuous-time devices whose behavior is accurately approximated in discrete time. Computers are designed in this way partly to minimize synchronization problems. Therefore, when a computer simulates a physical system – as opposed to solving analytically the equations that describe the system – you have to approximate the continuous-time system using the discrete-time behavior of the computer.

The leaky tank is perhaps the simplest interesting continuous-time systems and, as we find in later chapters, is a building block: an element from which to build complex systems. Similarly, its discrete-time approximation is the simplest interesting discrete-time system and is a building block for complex discrete-time systems. In this chapter we translate the leaky-tank model into continuous-time mathematics, use discrete-time approximations to understand its behavior qualitatively and quantitatively, and use the model to see why the water at the seashore is warmer in August than it is in June.

### 1.1 Leaky tank

Water flows into and leaves the leaky tank. We would like to describe the system mathematically. Then we can answer questions like ‘What happens if you dump a bit of water in: How fast does the water leak out and how does that rate change with time?’ This question may seem dull – what a simple input! – but keep your eyes on the prize: to understand this simple system because it is a building block for complex systems. One way to understand a system is to play with it: to try many inputs and to look at their outputs. So we play with it. As a precursor to trying various inputs, we need a mathematical description or model.
1.1.1 Mathematical model

To describe the system mathematically, we need a model of water leakage. The simplest model is that water leaks at a constant rate, independent of the height in the tank. That model is, however, too simple to show interesting behavior. So try the next-simplest model: that water leaks at a rate proportional to its level in the tank. This model is plausible because the pressure at the bottom of the tank is also proportional to the height. The model has the additional merit of generating interesting behavior without generating miserable mathematics.

To formalize this linear-leak model, introduce notation. Let \( r_1 \) be the rate at which water leaves the tank. Knowing the dimensions of all the quantities will help in later steps. So it is worth working out the following warmup:

**Pause to try 1.** What are the dimensions of \( r_1 \)?

These ‘pause to try’ questions are designed to help you read actively. Stop and think about the question, work out an answer, and then read onwards to compare your thoughts with the discussion in the text.

The rate \( r_1 \) has dimensions of volume per time. Now let’s look at other quantities and determine their dimensions. The other quantity that we need is the level in the tank; call it \( l \). The level has dimensions of length. With these symbols, the linear-leak model becomes

\[
r_1 \propto l \quad \text{(linear-leak equation)}.
\]

**Pause to try 2.** What are the dimensions of the missing constant of proportionality, which is hidden in the \( \propto \) symbol?

The constant of proportionality incorporates the fluid dynamics of the leak (for example, is it viscous or turbulent flow?) and it has dimensions of area per time.

The other part of the model is water conservation. The water that enters either leaves the tank or raises the level. There is no other place for the water to go. Therefore the difference between inflow and outflow changes the water level. Let \( r_0 \) be the rate at which water flows into the tank; like \( r_1 \), it has dimensions of volume per time. The level in the tank changes because of the net inflow:

\[
\dot{l} \propto r_0 - r_1 \quad \text{(water-conservation equation),}
\]

where the notation \( \dot{l} \) is a shorthand for \( dl/dt \). The missing constant of proportionality incorporates information about the dimensions of the tank. Here are a few questions for you to check your understanding. Unlike the ‘pause to try’ questions, the exercises do not have an answer in the following text. They might extend the analysis in the text, or be useful in others way to develop understanding.

**Exercise 1.** What are the dimensions of the missing constant of proportionality in the water-conservation equation?
Exercise 2. How does increasing the depth of the tank change the constant, if at all?

Exercise 3. How does increasing the width of the tank change the constant, if at all?

Now combine the water-conservation and linear-leak equations. The quickest method is to differentiate the linear-leak equation to get

\[ r_1 \propto \dot{l}. \]

The water-conservation equation expresses \( \dot{l} \) in terms of the inflow and outflow rates, so use that equation to eliminate the level \( l \). The result is

\[ r_1 \propto r_0 - r_1. \]

Pause to try 3. What are the dimensions of the missing constant of proportionality?

The missing constant has dimensions of inverse time, to match the \( dt \) in the denominator of \( r_1 \). Call the constant \( 1/\tau \). Then \( \tau \) is called the **time constant** of the system. With the \( \tau \) the leaky-tank **differential equation** is

\[ \tau \dot{r}_1 = r_0 - r_1 \]

or

\[ \dot{r}_1 = -\frac{r_1}{\tau} + \frac{r_0}{\tau}. \]

### 1.1.2 Analogous circuit

You have met this differential equation in previous courses. For example, here is an \( RC \) circuit. Its differential equation is

\[ \dot{V}_{out} = -\frac{V_{out}}{\tau} + \frac{V_{in}}{\tau}, \]

where the time constant \( \tau \) is \( RC \).

Exercise 4. Show that this differential equation is correct.

The \( RC \) equation has the same structure as the leaky-tank equation: The inflow \( r_0 \) is like the input voltage \( V_{in} \), and outflow \( r_1 \) is like the output voltage \( V_{out} \). Both equations also have a time constant \( \tau \), although each constant arises from physics particular to that situation. The time constant in the \( RC \) circuit arises from Maxwell’s equations of electromagnetism whereas the time constant in the leaky tank arises from the Navier–Stokes equations of fluid mechanics. With that caveat and by making the substitutions between flow rate and voltage, the two systems are identical.
By making this kind of analogy, you can use experience with circuits to help you understand mechanical systems and can use your experience with mechanical systems to understand circuits.

1.2 Qualitative understanding

Now that the system is represented as a differential equation, we can play with it to find out how the equation behaves – to obtain a qualitative understanding. We do so by using a simple input and by analyzing the resulting output using a discrete-time approximation simple enough to make with pencil and paper.

1.2.1 Simple input

You can qualitatively understand its behavior by reasoning through putting in a simple input signal \( r_0 \), hoping that the output signal \( r_1 \) will be easy to understand. Perhaps the simplest input is a step function. A step-function input means the input tap has been off forever, turns on at time \( t = 0 \), and remains at this flow rate \( r \) forever. To sketch the output, which is the flow rate \( r_1(t) \), use the useful technique of extreme cases. Here look at extreme cases of time. Typical extreme cases are \( t = -\infty, 0, \) and \( \infty \). Nothing of interest happens at \( t = -\infty \), so start investigating the next extreme, \( t \approx 0 \), and find the state of the system. Just at \( t = 0 \), when the tap turns on, the tank is still empty.

Pause to try 4. Why is the tank empty at \( t = 0 \)?

Since no water has flowed in since \( t = -\infty \), all the water in the tank has had time to drain. With zero pressure, the outflow rate \( r_1(0) \) is also zero. In this \( t \approx 0 \) limit, the \( r_1 \) term vanishes from the right side of the leaky-tank differential equation, which simplifies to:

\[
\dot{r}_1 = \frac{r_0}{\tau} \quad (t \approx 0).
\]

We want to solve for \( r_1 \), which is not hard because \( r_0 \) is a constant \( r \) for \( t \geq 0 \). The solution is linear growth:

\[
r_1(t) = \frac{t}{\tau} r,
\]

where \( r \) is the constant inflow rate. Although the outflow rate increases linearly in the \( t \approx 0 \) approximation, it cannot do so forever. Otherwise the outflow would eventually – when \( t > \tau \) – exceed the inflow, which would eventually drain the tank. Since an empty tank has no pressure and no outflow yet the outflow is supposed to be greater than \( r \), the situation contradicts itself. So the linear-increase approximation eventually refutes itself. However, like many things in life, the approximation is useful in moderation.

1.2.2 Discrete-time approximation

Rather than waiting for \( t > \tau \), we can use the approximation for a short time step \( \Delta t \). To decide how long \( \Delta t \) should be, think about the tradeoffs. The shorter the \( \Delta t \), the more accurate is the approximation to the continuous-time system. However, the longer the \( \Delta t \), the easier are the
paper-and-pencil calculations. We compromise between these opposing arguments by using $\Delta t = \tau / 2$.

In this approximation, $r_1$ starts at 0 and builds to $r_1(\tau / 2) = r / 2$. Then use the leaky-tank differential equation to find the new $\dot{r}_1$ at $t = \tau / 2$. The differential equation is

$$\dot{r}_1 = -\frac{r_1}{\tau} + \frac{r_0}{\tau}.$$  

For the second time step, which is the range $t = \Delta t \ldots 2\Delta t$, replace $r_1$ on the right side by its approximate value $r / 2$. As usual, $r_0$ on the right side is replaced by its steady value $r$. Then

$$\dot{r}_1 = -\frac{1}{2} \frac{r}{\tau} + \frac{r}{\tau} = \frac{1}{2} \frac{r}{\tau}.$$  

So $r_1$ increases at one-half of the steady rate that it increased in the first time step. Let’s run the machine for another time step. By the end of that step, $r_1$ reaches $3r / 4$. Following the same procedure, you find that the new $\dot{r}_1$ is $r / 4\tau$. Hmmm! The outflow is approaching the constant $r$ and it is increasing ever more slowly. In fact, the gap $r - r_1(t)$ halves with every time step $\Delta t = \tau / 2$ but never reaches zero. Here is a sketch of the outflow in this approximation:

![Sketch of the outflow in approximation](image)

The exact, continuous-time solution for a step-function input is

$$r_1(t) = \begin{cases} 0 & \text{for } t < 0; \\ r(1 - e^{-t/\tau}) & \text{for } t \geq 0. \end{cases}$$  

**Exercise 5.** Check this solution.

Here is a sketch of the exact solution:

![Sketch of the exact solution](image)
Pause to try 5. Compare the discrete-time, approximate solution and the continuous-time solution: Which converges more rapidly to the $t \to \infty$ value $r_1(\infty) = r$?

To compare the approximations, look at their respective values when $t = \tau$. Actually, look at their deviation from the final value $r_1(\infty) = r$. At time $\tau$, the continuous-time solution deviates by $r/e$, whereas this discrete-time solution deviates by $r/4$. Since $4 > e$, the discrete-time solution overestimates the rate of approach to the steady-state value. However, that inaccuracy is a worthwhile tradeoff to make in a first analysis. With the discrete-time approximation, we can discover important features of the solution using only paper, pencil, and sketches – items that would be available on a desert island.

1.3 Formalizing discrete-time: forward Euler

Let’s formalize the discrete-time approximation of the continuous-time differential equation. The steps to determine the next value of $r_1$ are:

1. Use the already computed $r_1(t)$ and the known input $r_0(t)$ to find $\dot{r}_1(t)$:

$$\dot{r}_1(t) = \frac{r_0(t) - r_1(t)}{\tau}.$$  

2. Assume that $\dot{r}_1(t)$ remains constant for the discretization time $\Delta t$ (in the previous analysis, $\Delta t = \tau / 2$).

3. Use that assumption, along with the already computed $r_1(t)$, to find $r_1(t + \Delta t)$:

$$r_1(t + \Delta t) = r_1(t) + \frac{\Delta t}{\tau} (r_0(t) - r_1(t)).$$

4. Go to step 1 with $t + \Delta t$ becoming the new $t$.

The discrete-time equation is

$$r_1(t + \Delta t) = \left(1 - \frac{\Delta t}{\tau}\right) r_1(t) + \frac{\Delta t}{\tau} r_0(t).$$

This equation is a difference equation and is an application of the forward-Euler method. It provides a way for a computer to solve the differential equation approximately. This method also provides a recipe that we can interpret to describe the behavior of the leaky tank. When $\Delta t \approx 0$, the first term on the right is nearly $r_1(t)$: It has weight $1 - \epsilon$ where $\epsilon = \Delta t / \tau$. The second term incorporates $r_0(t)$ with weight $\epsilon$. So the two terms combine into a weighted average of $r_1(t)$ and $r_0(t)$ with weights $1 - \epsilon$ and $\epsilon$, respectively. Therefore the leaky tank, like the RC circuit, outputs a decaying average of its input, which smooths the input. As an example, when the input is the step function, which has a discontinuity, the output is continuous, which is one derivative smoother than the step function.

1.3.1 Extreme cases of the time step

The hand simulation of the step-function response used $\Delta t = \tau / 2$. Let’s investigate the discrete-time approximation for other values of $\Delta t$.

First try $\Delta t = \tau$. The outflow $r_1(t)$ is a straight ramp till $r_1(t) = r$, which happens at the end of the time step when $t = \tau$. Since $r_0(t) = r$ for $t > 0$, the difference between inflow and outflow is zero, so the level does not change, and the outflow is also constant. Thus the approximate solution is:
Now increase $\Delta t$ to say $\Delta t = 2\tau$. The same initial ramp happens as when $\Delta t = \tau$, but the ramp now ends later at $t = 2\tau$. Then $r_1(2\tau) = 2r$. The leaky-tank differential equation then gives $r_1(2\tau) = -r/\tau$. Applying this derivative for one time step $\Delta t = 2\tau$ produces a new flow rate $r_1(2\Delta t) = 0$. From here the cycle repeats. This barely stable oscillation is interesting because the exact solution shows no oscillation. So with $\Delta t = 2\tau$, the discrete-time approximation produces a qualitatively incorrect conclusion.

Now increase $\Delta t$ to $3\tau$. The values of $r_1$ are:

\[
\begin{align*}
r_1(0\Delta t) & = 0, \\
r_1(1\Delta t) & = 3r, \\
r_1(2\Delta t) & = -3r, \\
r_1(3\Delta t) & = 9r, \\
r_1(4\Delta t) & = -15r, \\
\ldots
\end{align*}
\]

These oscillations are unstable! This discrete-time approximation gives qualitatively incorrect results, for example suggesting that the water level in the tank, or the rate of outflow, can oscillate unstably despite a steady input. But as we will see later, it takes at least two such systems connected with feedback to create oscillations.

These qualitative inaccuracies suggest trying the other extreme of $\Delta t$, that $\Delta t \to 0$. To find how the system behaves, return to the difference equation:

\[
r_1(t + \Delta t) = \left(1 - \frac{\Delta t}{\tau}\right)r_1(t) + \frac{\Delta t}{\tau}r_0(t).
\]

It is easiest to solve if we make a few changes. First, define

\[
y[n] \equiv r_1(n\Delta t).
\]

The time-step number $n$ is a dimensionless measure of time. With $r_0(t) = r$, the leaky-tank difference equation is

\[
y[n + 1] = (1 - \epsilon)y[n] + \epsilon r,
\]

where $\epsilon = \Delta t/\tau$. Now make a dimensionless measure of rate: $Q = Y/r$. This division is an example of a useful technique: taking out the big part. Then

\[
q[n + 1] = (1 - \epsilon)q[n] + \epsilon.
\]
Now take out the big part again by measuring the rate relative to its long-term, steady-state value of $r$, which here is $q[\infty] = 1$. So define $P = 1 - Q$. Then
\[ 1 - p[n + 1] = (1 - e)(1 - p[n]) + \epsilon, \]

or
\[ p[n + 1] = (1 - e)p[n]. \]

With the boundary condition $p[0] = 1$, the solution is
\[ p[n] = (1 - e)^n. \]

Invert all the changes of variable to solve for $r_1(t)$:
\[ r_1(n\Delta t) = r (1 - (1 - e)^n). \]

Now use the approximation that $\Delta t \to 0$, so $e = \Delta t/\tau \to 0$ as well. Then
\[ (1 - e)^n \approx e^{-ne}. \]

**Exercise 6.** Justify this approximation for $e^x$.

In this $e \to 0$ limit,
\[ r_1(n\Delta t) = (1 - e^{-n\Delta t/\tau})r. \]

Call $t = n\Delta t$ and get
\[ r_1(t) = (1 - e^{-t/\tau})r, \]

which is the exact solution from integrating the continuous-time, differential equation.

### 1.4 Heat storage in the ocean

An ocean is a leaky-tank system! Heat flows into the ocean from the sun, land, and air. Heat inflow is proportional to the land or air temperature (leaving out the sun for simplicity). Heat leaks from the ocean into the surroundings at a rate proportional to the ocean temperature. So the ocean temperature is like the tank level; the input signal is the land or air temperature, which we approximate as the same; and the output signal is the ocean temperature. With this analogy, we can simulate the water temperature by using the leaky-tank equations.

The only parameter of a leaky-tank system is its time constant. The ocean’s time constant is perhaps $\tau = 3$ months, because it can store a lot of heat. The input temperature $T_0(t)$, produced by the ground and air warmed by the sun, oscillates with a period of 6 months.

**Pause to try 6.** Simulate the system to find the output temperature $T_1(t)$ and discuss any interesting features of the result.

To simulate the output temperature $T_1(t)$, use the forward-Euler method. Here is a simulation for when the input signal is zero for $t < 0$ and starts oscillating at $t = 0$: 
The output oscillates, which is surprising. When you look at the equation, it is not obvious that a sinusoidal input will give a sinusoidal output. But the oscillating output makes sense if you have swum at the beach during several seasons: winter is colder than spring, which is colder than summer, etc.

We can also find the temperature by solving the differential equation in closed form. When the system is given an eternal sinusoidal input $T_0(t) = A \sin \beta t$ (including for $t < 0$), the closed-form solution is

$$T_1(t) = \frac{A}{\sqrt{1 + (\beta \tau)^2}} \sin(\beta t - \phi),$$

where $\phi = \arctan \beta \tau$ is the so-called phase lag.

Exercise 7. Check this solution.

This form shows two features of the output temperature:

- **diminution**: Because of the $\sqrt{1 + (\beta \tau)^2}$ factor in the denominator, which is greater than 1, the output signal is shrunken compared to the input signal. Therefore ocean temperature fluctuates less than land temperature, and the ocean is more hospitable for life to evolve.

- **delay**: Because of the $-\phi$ contribution to the phase, the output is delayed relative to the input. So, although the sun’s peak temperature input is in June at the summer solstice, the ocean temperature peaks a few months later. The best time to swim or surf in the Massachusetts waters is in August!

These two behaviors, which are canonical for a first-order system such as a tank, will be revisited when we study signal processing and we will build intuitions for each behavior.

Exercise 8. Why does an unheated house feel coldest at around, say, 4am rather than at midnight, even though more sunlight is available at 4am to heat the ground and house?
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