Last Time: System Functions

Substituting \( \frac{1}{s} \) for \( A \) and \( s \) for \( D \) converts a CT system functional into its corresponding **system function** \( H(s) \).

\[
H(s) = \frac{Y}{X} \bigg|_{A \rightarrow \frac{1}{s}, D \leftarrow s}
\]

Poles (or zeros) are roots of the denominator (or numerator) polynomial of \( H(s) \).

\[
H(s) = \frac{Y}{X} = \frac{100s}{s^2 + 2s + 100}
\]

Last Time: Eigenfunctions and Eigenvalues

The system function can be used to determine response of a system to exponential inputs.

If \( x(t) = e^{s_0 t} \) (for all time) then \( y(t) = H(s_0)e^{s_0 t} \) (for all time).

Exponentials are eigenfunctions of systems that can be represented by linear differential equations with constant coefficients.

The eigenvalue associated with the eigenfunction \( e^{s_0 t} \) is the value of the system function \( H(s) \) at \( s = s_0 \).
The value of \( H(s_0) \) at a point \( s = s_0 \) can be determined graphically using vector analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

\[
H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots}
\]

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here \( z_0 \)) to \( s_0 \), the point of interest in the \( s \)-plane.

The magnitude of \( H(s_0) \) is the product of the magnitudes of the vectors associated with the zeros divided by the product of the magnitudes of the vectors associated with the poles.
**Last Time: Vector Diagrams**

The angle of $H(s_0)$ is the sum of the angles of the vectors associated with the zeros minus the sum of the angles of the vectors associated with the poles.

$$\angle H(s_0) = \angle \left( \prod_{q=1}^{Q} \frac{s_0 - z_q}{s_0 - \omega} \right) = \angle \left( K + \sum_{q=1}^{Q} \angle \left( s_0 - z_q \right) - \sum_{p=1}^{P} \angle \left( s_0 - p_p \right) \right)$$

The angle of $K$ can be 0 or $\pi$ for systems described by linear differential equations with constant, real-valued coefficients.

**Last Time: Frequency Response**

The frequency response of a system is given by $H(j\omega)$.

If $x(t) = \cos \omega_0 t$ then

$$y(t) = |H(j\omega)| \cos (\omega_0 t + \angle(H(j\omega)))$$

The frequency response is a complex-valued function of $\omega$. The magnitude gives the gain of the system for each frequency. The angle gives the phase.
Frequency Response: $H(s)|_{s \rightarrow j\omega}$

\[ H(s) = \frac{9}{s - p_1} \]

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]

Check Yourself

Could the phase plots of any of the systems represented by the following pole-zero plots be equal to each other? [caution: this could be a trick question]
From Frequency Response to Bode Plot

The magnitude of $H(jω)$ is a product of magnitudes.

$$|H(jω)| = |K| \prod_{q=1}^{Q} \frac{|jω - z_q|}{|jω - p_p|}$$

The angle of $H(jω)$ is a sum of angles.

$$∠H(jω) = ∠K + \sum_{q=1}^{Q} ∠(jω - z_q) - \sum_{p=1}^{P} ∠(jω - p_p)$$

The angle of $K$ can be 0 or $π$ for systems described by linear differential equations with constant, real-valued coefficients.

From Frequency Response to Bode Plot

The log of the magnitude is a sum of logs.

$$\log |H(jω)| = \log |K| + \sum_{q=1}^{Q} \log |jω - z_q| - \sum_{p=1}^{P} \log |jω - p_p|$$

$$∠H(jω) = ∠K + \sum_{q=1}^{Q} ∠(jω - z_q) - \sum_{p=1}^{P} ∠(jω - p_p)$$

Bode Plot: Isolated Zero

The two asymptotes are a good approximation to $\log |H(jω)|$.

$$H(s) = s - z_1 , \quad z_1 < 0$$

$$\lim_{ω→0} |H(jω)|/|z_1|$$

$$\lim_{ω→∞} |H(jω)| = |z_1|$$

$$\lim_{ω→∞} |H(jω)| = ω$$
**Bode Plot: Isolated Zero**

Straight-line approximation to $\angle H(j\omega)$.

$H(s) = s - z_1$, $z_1 < 0$

\[
\lim_{{\omega \to 0}} \angle H(j\omega) = 0 \\
\lim_{{\omega \to \infty}} \angle H(j\omega) = \pi/2
\]

**Bode Plot: Isolated Pole**

The two asymptotes are a good approximation to $\log |H(j\omega)|$.

$H(s) = \frac{1}{s - p_1}$, $p_1 < 0$

\[
\lim_{{\omega \to 0}} |H(j\omega)| = \frac{1}{|p_1|} \\
\lim_{{\omega \to \infty}} |H(j\omega)| = \frac{1}{\omega}
\]

**Bode Plot: Isolated Pole**

Straight-line approximation to $\angle H(j\omega)$.

$H(s) = \frac{1}{s - p_1}$, $p_1 < 0$

\[
\lim_{{\omega \to 0}} \angle H(j\omega) = 0 \\
\lim_{{\omega \to \infty}} \angle H(j\omega) = -\pi/2
\]
**Check Yourself**

\[ H_1(s) = \frac{1}{s + 1} \quad \text{and} \quad H_2(s) = \frac{10}{s + 10} \]

The Bode magnitude plot for \( H_2(s) \) can be obtained from that for \( H_1(s) \) by

1. shifting it horizontally
2. scaling it horizontally
3. shifting and scaling it horizontally
4. shifting and scaling both horizontally and vertically
5. none of the above

**Bode Plot: More Complicated**

\[ H(s) = \frac{s}{(s + 1)(s + 10)} \]

![Bode Plot Diagram](image)
Bode Plot: More Complicated

\[ H(s) = \frac{s}{(s + 1)(s + 10)} \]

\[ \log |H(j\omega)| \]

\[ \angle H(j\omega) \]
Bode Plot: dB

\[ H(s) = \frac{10s}{(s + 1)(s + 10)} \]

Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.

\[ H(\omega) = \frac{1}{j\omega + 1} \]

Check Yourself

Without using a calculator (other than the one in your head) determine

1. the frequency that is 6 dB below 1 kHz
2. the amplitude that is 10 dB above 1
3. the value of 500 in dB
4. how many dB/octave correspond to a slope of 20 dB/decade
Frequency Response of a High-\(Q\) System

The magnitude of the frequency response of a high-\(Q\) system is peaked.

\[
H(s) = \frac{1}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2}
\]

\[
\text{s-plane}
\]

\[
\text{log} |H(j\omega)|
\]

\[
\omega/\omega_0
\]

[log scale]

Check Yourself

Estimate the peak value of the magnitude function as a function of \(Q\) assuming \(Q\) is large (e.g., \(Q > 3\)).

\[
H(s) = \frac{1}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2}
\]
Frequency Response of a High-Q System

As $Q$ increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Check Yourself

Estimate the “3dB bandwidth” of the peak.

Let $\omega_l$ (or $\omega_h$) represent the lowest (or highest) frequency for which the magnitude is greater than the peak value divided by $\sqrt{2}$. The 3dB bandwidth is then $\omega_h - \omega_l$.

Frequency Response of a High-Q System

As $Q$ increases, the phase changes more abruptly with $\omega$.
**Frequency Response of a High-Q System**

As \( Q \) increases, the phase changes more abruptly with \( \omega \).

\[
H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}
\]

\[\frac{s}{\omega_0}-\text{plane}\]

\[
\angle H(j\omega) = -\arctan\left(\frac{\omega}{\omega_0}\right)
\]

\[\omega/\omega_0\]

---

**Check Yourself**

Estimate the change in phase that occurs over the 3dB bandwidth.

\[
H(s) = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}
\]

---

**Multiple Representations of CT Systems**

**Verbal descriptions:** preserve the underlying physics.

**Differential equations:** mathematically compact.

\[
\dot{y}(t) = x(t) + py(t)
\]

**Block diagrams:** illustrate signal flow paths.

**Operator representations:** analyze systems as polynomials.

\[
(1 - p\mathcal{A})Y = \mathcal{A}X
\]

**Pole-Zero diagrams:** represent factors of system functional.

**System functions:** represent systems as polynomials in \( s \).

**Bode plots:** quickly sketch the frequency response.