Lecture 14
CT Feedback and Control
Feedback and Control

Feedback is pervasive in natural and artificial systems.

Turn steering wheel to stay centered in the lane.
Feedback and Control

Feedback is useful for regulating a system’s behavior, as when a thermostat regulates the temperature of a house.

![Thermostat Diagram]

- Desired temperature
- Thermostat
- Heating system
- Actual temperature

Diagram: A house with a thermometer indicating temperature $T$. The desired temperature flows into the thermostat, which controls the heating system, and the actual temperature is monitored and fed back to the thermostat for adjustment.
Feedback and Control

Concentration of glucose in blood is highly regulated and remains nearly constant despite episodic ingestion and use.
Feedback and Control

Motor control relies on feedback from pressure sensors in the skin as well as proprioceptors in muscles, tendons, and joints.

Try building a robotic hand to unscrew a lightbulb!

Shadow Dexterous Robot Hand (Wikipedia)
Feedback and Control

We’ve previously used systems theory to gain insight into how to control a discrete-time system.

Today’s goal: use systems theory to gain insight into how to control a continuous-time system.
Example: Op Amp

Feedback is often used to improve the performance of a system as characterized by some figure of merit.

Example: Design an amplifier for processing audio signals using an op amp.

An op amp is a device with 2 inputs ($V_+$ and $V_-$) and one output ($V_o$) which is approximately equal to a gain ($F(s)$) times the difference between the inputs ($V_+ - V_-$).

Op Amp:

\[
V_+ \quad + \quad F(s) \quad V_o = F(s) (V_+ - V_-)
\]
Example: Op Amp

Op amps have high gain at low frequencies, but the gain decreases with frequency.

We can represent the frequency dependence of a typical op amp (LM741) with a system function of the form

\[ F(s) = \frac{F_0}{1 + \tau s} \]

where \( F_0 \approx 2 \times 10^5 \) and \( \tau \approx 1/40 \) seconds.
Example: Op Amp

Without feedback, the LM741 is a poor audio amplifier.

\[ F(s) = \frac{F_0}{1 + \tau s} ; \quad F_0 \approx 2 \times 10^5 ; \quad \tau \approx 1/40 \text{ seconds} \]
Example: Op Amp

Without feedback, the LM741 is slow.

Impulse response

\[ y(t) = \frac{F_0}{\tau} e^{-t/\tau}, \quad t > 0 \]

Step response

\[ s(t) = F_0 (1 - e^{-t/\tau}), \quad t > 0 \]

Fast for humans ... slow for electronics!
Example: Op Amp

Use feedback to improve the frequency response and/or the speed of an op amp.

\[ V_\text{in} = \beta V_\text{out} = \left( \frac{R_2}{R_1 + R_2} \right) V_\text{out} \]

\[ \frac{V_\text{out}}{V_\text{in}} = \frac{F(s)}{1 + \beta F(s)} \]

\[ = \frac{F_0}{1 + \beta F_0} \cdot \frac{1}{1 + \frac{s\tau}{1 + \beta F_0}} \]
What is the most negative value of the closed-loop pole that can be achieved with feedback?
Check Yourself: Op Amp

With feedback, the closed-loop pole is at

\[ s = -\frac{1 + \beta F_0}{\tau}. \]

The feedback constant is \( 0 \leq \beta \leq 1 \). Therefore the most negative value of the closed-loop pole is \((1 + F_0)/\tau\).

Feedback shifts the pole to the left by a factor as large as \(1 + F_0 = 2 \times 10^5\).
Feedback extends the frequency response by as much as a factor of $F_0 = 2 \times 10^5$. 
Example: Op Amp

Feedback produces higher bandwidths by reducing the gain at low frequencies. It trades gain for bandwidth.
Example: Op Amp

Feedback makes the time response faster by as much as a factor of \( F_0 = 2 \times 10^5 \).

Step response

\[
s(t) = \frac{F_0}{1 + \beta F_0} \left(1 - e^{-t/\left(\tau/(1+\beta F_0)\right)}\right), \quad t > 0
\]
Example: Op Amp

Feedback produces faster responses by **reducing** the final value of the step response. It trades gain for speed.

**Step response**

\[ s(t) = \frac{F_0}{1 + \beta F_0} \left(1 - e^{-t/(\tau/(1+\beta F_0))}\right), \quad t > 0 \]
Example: Op Amp

Summary

- feedback can extend frequency response
- feedback can increase speed

These performance enhancements are achieved through a reduction of gain.
Feedback and Control

Feedback can also be used to control a system.

Black’s equation for the closed-loop gain $H(s)$:

$$
\frac{Y}{X} = H(s) = \frac{K(s)F(s)}{1 + K(s)F(s)G(s)}
$$

- **forward gain**
- **loop gain**
Example: Motor Controller

We wish to build a robot arm (actually its elbow). The input should be voltage $v(t)$, and the output should be the elbow angle $\theta(t)$.

$$v(t) \rightarrow \text{robotic arm} \rightarrow \theta(t) \propto v(t)$$

We wish to build the robot arm with a DC motor.

$$v(t) \rightarrow \text{DC motor} \rightarrow \theta(t)$$
Example: Motor Controller

What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

$v(t) \rightarrow \text{DC motor} \rightarrow \theta(t)$
Example: Motor Controller

To first order, the rotational speed $\dot{\theta}(t)$ of a DC motor is proportional to the input voltage $v(t)$.

\[ v(t) \rightarrow \text{DC motor} \rightarrow \theta(t) \]

First-order model: integrator
Example: Motor Controller

Use proportional feedback to control the angle of the motor’s output.

\[ \Theta = \frac{\alpha \gamma \mathcal{A}}{1 + \alpha \beta \gamma \mathcal{A}} = \frac{\alpha \gamma \frac{1}{s}}{1 + \alpha \beta \gamma \frac{1}{s}} = \frac{\alpha \gamma}{s + \alpha \beta \gamma} = \frac{1}{\beta} \]
Example: Motor Controller

The closed loop system has a single pole at $s = -\alpha \beta \gamma$.

$$\frac{\Theta}{V} = \frac{\alpha \gamma}{s + \alpha \beta \gamma} = \frac{1/\beta}{1 + s/(\alpha \beta \gamma)}$$

If the gain $\alpha$ is zero, then the closed loop pole is at $s = 0$ (i.e., same as open loop).

If the gain increases, the closed loop pole moves to the left.
Example: Motor Controller

Find the step response by integrating the impulse response.

\[
\frac{\Theta}{V} = \alpha \gamma \frac{\mathcal{A}}{1 + \alpha \beta \gamma \mathcal{A}}
\]

If \( v(t) = \delta(t) \) then \( \theta(t) = \alpha \gamma e^{-\alpha \beta \gamma t} ; \ t \geq 0. \)

If \( v(t) = u(t) = \int_{-\infty}^{t} \delta(t') \, dt' \) then

\[
\theta(t) = \int_{-\infty}^{t} \alpha \gamma e^{-\alpha \beta \gamma t'} u(t') \, dt' = \int_{0}^{t} \alpha \gamma e^{-\alpha \beta \gamma t'} \, dt' = \frac{1}{\beta} \left( 1 - e^{-\alpha \beta \gamma t} \right) ; \ t \geq 0.
\]

The response is faster for larger values of \( \alpha \).

Try it: Demo.
Example: Motor Controller

The speed of a DC motor does not change instantly if the voltage is stepped. There is lag due to rotational inertia.

Second-order model: integrator with lag

\[
V \rightarrow \gamma A \left( \frac{p A}{1 + p A} \right) \rightarrow \Theta
\]

Step response:

\[
\theta(t) = v_{step} \left( \gamma t - \frac{\gamma}{p} (1 - e^{-pt}) \right) \quad \text{for } t \geq 0
\]
Example: Motor Controller

Analyze second-order model.

\[
\Theta = \frac{\alpha \gamma p A^2}{1 + pA} = \frac{\alpha \gamma p A^2}{1 + pA + \alpha \beta \gamma p A^2} = \frac{\alpha \gamma p}{s^2 + ps + \alpha \beta \gamma p}
\]

\[
s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - \alpha \beta \gamma p}
\]
Example: Motor Controller

For second-order model, the closed-loop poles are at 0 and $-p$ when $\alpha = 0$. Increasing $\alpha$ causes the poles to approach each other, collide at $s = -p$ then split into two poles with imaginary parts.
Feedback and Control: Summary

CT feedback is useful for many reasons. Today we saw two:

1. to improve the performance of a system (op amp) as characterized by a figure of merit (e.g. bandwidth, speed), and

2. to control a system (DC motor) so that the output is position rather than speed.