After studying this chapter, you should be able to:

- switch between the functional for a system (or the system function) and the signal representation;
- sketch the convolution of two signals; and
- compute the convolution of two signals analytically.

The theme of this course is *multiple representations*. Every fortnight we throw another representation at you. The newest representation for a system is by its impulse response. This representation, like any representation, makes some operations easier and others harder. It was easy to compose systems when representing a system by its functional or system function: Simply multiply the functions or functionals. In the impulse-response representation, the operation of convolution performs the same task. Convolution is not as simple as multiplication is, so in this chapter we develop intuitions for how it behaves.

But first a word about notation. The step function $u(t)$ often clutters integrals and other formulas, making it hard for me to think. Therefore, most formulas will omit the $u(t)$ if the context makes the meaning clear without $u(t)$. For example, the chapter might say that the impulse response of an RC circuit or leaky tank with $\tau = 1$ is $e^{-t}$, when the full story is that the impulse response is $e^{-t}u(t)$.

### 17.1 Convolution is a weighted average

The fundamental intuition about convolution is that the convolution $f \ast g$ computes a weighted average of $f$ where the weights are given by $g$. As an extreme example, take $g$ to be the impulse $\delta(t)$, and $f$ to be the impulse response of a leaky tank in units where the time constant $\tau = 1$, so $f(t) = e^{-t}$ for $t \geq 0$.

**Pause to try 59.** What is $f \ast g$?

The weight function $g$ is an impulse, which has all its weight at 0. Therefore the weighted average of $f$ is just a copy:

$$e^{-t} \ast \delta(t) = e^{-t}.$$  

That intuitive argument leads to a general result: Convolving any function $f$ with an impulse reproduces $f$. 

Let’s check the specific result in two ways. The first way is from the origin of convolution as a way to compose systems. In the next figure, the dashed lines indicate that the signal $F$ is a representation for the system with system function $H_F(s)$.

The signal $F$ is the impulse response of a leaky tank with system function $1/(1 + s)$. The signal $\delta(t)$ is the impulse response of a wire, which has system function 1. So the previous figure becomes

The cascade has system function

$$\frac{1}{1 + s} \times 1 = \frac{1}{1 + s}.$$ 

Its impulse response is $e^{-t}$ so

$$e^{-t} \star \delta(t) = e^{-t}.$$ 

This systems method generalizes to the same conclusion: Convolving any function $f$ with an impulse reproduces $f$.

The second way to check the intuitive result is by doing integrals. The principle of extreme laziness recommends avoiding integrals until you have tried every other method, wherefore we integrate only now. The convolution integral is

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) \, d\tau.$$ 

With $g(t) = \delta(t)$, the integral is

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau) \, d\tau.$$ 

The delta function is nonzero only when $t = \tau$, so the delta function picks out $f(t)$:

$$\int_{-\infty}^{\infty} f(\tau)\delta(t - \tau) \, d\tau = f(t).$$ 

This result, that convolving with a delta function does nothing, is the general result that we stated but did not quite prove by the averaging argument or by the systems method.

### 17.2 Convolving with a delayed impulse

Let’s now try the next-simplest convolution example: Convolve $f(t)$ with the shifted impulse $\delta(t - t_0)$. The averaging method suggests that the result will be like $f$, perhaps with a shift. But
the method does not make it obvious which direction to shift. So let’s try the systems method. A shifted impulse corresponds to a wire with a delay, which is what the $R$ operator does. In the discrete-time chapters, the $R$ operator delayed a signal by the time step $T$. To make the amount of delay explicit, we subscript the $R$ operator with the delay. Then $g(t) = \delta(t - t_0)$ is represented by the system $R_{t_0}$. The convolution $f \star g$ is then $R_{t_0} f(t) = f(t - t_0)$. This argument is represented by the following systems analysis:

So convolving $f$ with the delayed impulse, which has all its weight at one point, reproduces $f$ but with a delay. Check this result by doing the convolution integral. The integral is

$$ (f \star g)(t) = \int f(\tau) \delta(t - \tau - t_0) \, d\tau, $$

which is $f(t - t_0)$ because the delta function picks out the $f(\tau)$ where $\tau = t - t_0$.

17.3 Two leaky tanks

Next convolve $e^{-t}$ with a weight function that is more interesting than $\delta(t)$ or, if it is possible, is more interesting than $\delta(t - t_0)$. In particular, we convolve $e^{-t}$ with itself and compute $e^{-t} \star e^{-t}$.

17.3.1 Qualitative analysis

Follow Wheeler’s principle: Figure out the characteristics of the convolution before doing extensive calculations. The fundamental intuition is that convolution is a weighted average. In the extreme case of averaging $f$ using an infinitely thin function (a delta function), $f$ comes through unscathed. Convolving $f$ with an exponential decay, on the other hand, will smooth $f$. Since $f$ has a discontinuity, the smoothed version will probably have a discontinuity in its slope but itself be continuous. So at $t = 0$ the convolution should rise linearly from zero. Whereas for large $t$ the convolution should decay to zero: Since $f$ itself decays to zero, a smoothed version of $f$ should do the same.

The next intuition is about the delay, or time shift of the convolution. Our averaging function, an exponential decay, begins at $t = 0$ and extends to the right. Extent can reasonably be defined as how long it takes the function to fall significantly, and $e$ is a convenient choice for the significant factor. By this definition, $g$ extends one time unit to the right, so it has some features of the delayed impulse $\delta(t - 1)$. The signal $g$ is of course smoother than the delayed impulse, but when qualitatively working out the delay in $f \star g$, a delayed impulse is a useful approximation to $g$. 

$$
\begin{array}{c}
\text{0} \quad e^{-t} \quad \star \\
\end{array}
$$

$$
\begin{array}{c}
\text{0} \quad e^{-t} \\
\end{array}
$$
Then since $f(t)$ peaks at $t = 0$, convolving $f$ with $g$ will shift the peak right by one time unit. So $f \star g$ should peak when $t \approx 1$.

A related intuition is about the width and height of $f \star g$. Its width and height depend on the width and height of $f$ and $g$. We would like approximations for $f$ and $g$ that have easily tunable height and width and that are easy to reason with. A useful approximation is a pulse because we can qualitatively predict the result of convolving pulses. To choose the width and height of the pulse, notice that convolution multiplies areas:

**Exercise 78.** Show that

$$\text{Area of } f \star g = \text{Area of } f \times \text{Area of } g.$$  

So we should approximate $f$ (or $g$) using a pulse that has the same area as $f$. Since $f$ has unit area, the pulse’s width should be the reciprocal of its height. The height should be a typical or average height, where the average is computed by weighting the actual height more strongly where $f$ is more important (is larger). In this definition, the average height of $f$ will be less than the maximum height of $f$. A reasonable guess is that the typical height is $1/2$. So the width will be 2, producing the pulse approximation marked by the dotted line.

**Pause to try 60.** Sketch the convolution of the pulse with itself.

Convolving this pulse with itself produces a triangle. The triangle has a width (a base) of 4. You can see that result in two ways. You can do the integration, which was done in lecture. Or you can think qualitatively about convolution as averaging. Averaging $f$ using $g$ smears $f$ by the width of $g$. This smearing should produce a function whose width is the sum of the widths of $f$ and $g$. So the width of $f \star g$ should be 4. With a width of 4 and an area of 1, the triangle should have a height of $1/2$.

We have reasoned our way to several qualitative conclusions about $e^{-t} \star e^{-t}$. The convolution should

1. rise linearly from zero at $t = 0$,
2. peak near $t = 1$,
3. decay to zero for large $t$, and
4. have a base width of roughly 4 and peak height of roughly $1/2$.

Here is a sketch that is consistent with these conclusions:

Next check the conclusions by finding the exact convolution.
17.3.2 Systems analysis

One way to find the exact convolution is a systems analysis:

\[ e^{-t} \star e^{-t} = te^{-t}. \]

In pictures:

Let’s check the qualitative conclusions against the expression \( te^{-t} \) and its picture. The function rises linearly from zero at \( t = 0 \), as predicted. It decays to zero for large \( t \), also as predicted. It peaks at \( t = 1 \), as you can show by maximizing \( te^{-t} \) using differentiation. So the peak’s location is also as predicted. It has a peak height of \( 1/e \), so the qualitative estimate of \( 1/2 \) for the peak height is reasonably accurate, much more than we have a right to expect given the number of approximations that we made!

Exercise 79. Use systems analysis to show that \( f \star g = g \star f \) for any \( f \) and \( g \).

17.3.3 Integration

As a last resort, compute the convolution by integration. The integral is

\[ \int f(\tau)g(t - \tau)d\tau. \]

Rather than strew the integral with lots of step functions (the horrible \( u(t) \) notation), let’s just figure out the correct integration limits. The integrand is nonzero only when \( f(\tau) \) and \( g(t - \tau) \) are nonzero. Our signals are nonzero only for positive time. So the integrand contributes something only when \( \tau > 0 \) (to make \( f(\tau) \) nonzero) and \( t > \tau \) (to make \( g(t - \tau) \) nonzero). The two conditions restrict \( \tau \) to the range \((0, t)\), and make the integral

\[ \int_0^t e^{-\tau}e^{t-\tau}d\tau. \]

The \( e^{-\tau} \) factor that is part of \( e^{t-\tau} \) is a constant when doing a \( d\tau \) integral, so it can be pulled out. The remaining integrand is 1 integrated over a range of length \( t \), so the convolution is
\[ f \star g = \int_0^t e^{-\tau} e^{t-\tau} d\tau = te^{-t}, \]

which agrees with the result from systems analysis.

### 17.4 Summary

The main message of this chapter is that you can sketch convolutions qualitatively before doing integrals. To make those sketches, several qualitative arguments are useful:

1. Convolution is a weighted average, and convolving (with a positive function) does a smoothing.

2. The width of \( f \star g \) is the sum of the widths of \( f \) and \( g \).

3. A delay in the averaging function (the \( g \)) delays the convolution \( f \star g \).

4. Convolution multiplies areas:

   \[ \text{Area of } f \star g = \text{Area of } f \times \text{Area of } g. \]

These qualitative arguments are particularly useful for understanding and designing complicated systems – for example, to understand the atmosphere and Hubble telescope lens together smear the image of the sky.