Lecture 16
Convolution
Multiple Representations of CT and DT Systems

Throughout the semester, we have seen a variety of representations of systems, and how they can each be useful.

Today we add yet another, this one based on the impulse response.
Cascading Systems

Cascade two systems → multiply their system functions and functionals.

Cascade:

\[ X \xrightarrow{H_1(s)} Y_1 \xrightarrow{H_2(s)} Y_2 \]

Equivalent representation:

\[ X \xrightarrow{H_1(s) \times H_2(s)} Y_2 \]
Cascading Systems

Example:

\[
\frac{Y_2}{X} = (1 + R + R^2) \times (1 + R + R^2)
\]

<table>
<thead>
<tr>
<th></th>
<th>1 + R</th>
<th>+R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>+R</td>
<td>R</td>
<td>R²</td>
</tr>
<tr>
<td>+R²</td>
<td>R²</td>
<td>R³</td>
</tr>
</tbody>
</table>

\[
\frac{Y_2}{X} = 1 + 2R^2 + 3R^3 + 2R^4 + R^5
\]
Cascading Systems

Example:

\[(1 + R + R^2) \times (1 + R + R^2) = 1 + 2R^2 + 3R^3 + 2R^4 + R^5\]

\[h_1[n]\]

\[h_2[n]\]

\[h[n]\]
Cascading Systems

Generalize to arbitrary causal DT systems.

\[
\frac{Y_1}{X} = a_0 + a_1 R + a_2 R^2 + \cdots \quad \frac{Y_2}{Y_1} = b_0 + b_1 R + b_2 R^2 + \cdots
\]

\[
\frac{Y_1}{X} = (a_0 + a_1 R + a_2 R^2 + \cdots) \times (b_0 + b_1 R + b_2 R^2 + \cdots)
\]

<table>
<thead>
<tr>
<th></th>
<th>(a_0)</th>
<th>(+a_1 R)</th>
<th>(+a_2 R^2)</th>
<th>(\cdots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_0)</td>
<td>(a_0b_0)</td>
<td>(a_1b_0R)</td>
<td>(a_2b_0R^2)</td>
<td>(\cdots)</td>
</tr>
<tr>
<td>(+b_1 R)</td>
<td>(a_0b_1R)</td>
<td>(a_1b_1R^2)</td>
<td>(a_2b_1R^3)</td>
<td>(\cdots)</td>
</tr>
<tr>
<td>(+b_2 R^2)</td>
<td>(a_0b_2R^2)</td>
<td>(a_1b_2R^3)</td>
<td>(a_2b_2R^4)</td>
<td>(\cdots)</td>
</tr>
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<td>(\cdots)</td>
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<td>(\cdots)</td>
<td>(\cdots)</td>
<td>(\cdots)</td>
</tr>
</tbody>
</table>

\[
\frac{Y_2}{X} = a_0b_0 + (a_0b_1 + a_1b_0)R + (a_0b_2 + a_1b_1 + a_2b_0)R^2 + \cdots
\]
Polynomial multiplication: collect terms with equal delays.

\[
\frac{Y}{X} = (1 + aR + a^2R^2 + a^3R^3 + \cdots) \times (1 + bR + b^2R^2 + b^3R^3 + \cdots)
\]

\[
\frac{Y}{X} = 1 + (a + b)R + (a^2 + ab + b^2)R^2 + (a^3 + a^2b + ab^2 + b^3)R^3 + \cdots
\]
Cascading Systems

Generalize to arbitrary causal DT systems.

\[
\frac{Y_2}{X} = a_0 b_0 + (a_0 b_1 + a_1 b_0)R + (a_0 b_2 + a_1 b_1 + a_2 b_0)R^2 + \cdots
\]

Let

\[
\frac{Y_2}{X} = c_0 + c_1 R + c_2 R^2 + \cdots
\]

Then

\[
c_n = \sum_{k=0}^{n} a_k b_{n-k}
\]
Cascading Systems

Generalize to non-causal DT systems.

\[
X \xrightarrow{L + 1 + R} Y_1 \xrightarrow{L + 1 + R} Y_2
\]

\[
\frac{Y_2}{X} = (L + 1 + R) \times (L + 1 + R)
\]

\[
\begin{array}{c|ccc}
 & L & +1 & +R \\
\hline
L & L^2 & L & 1 \\
+1 & L & 1 & R \\
+R & 1 & R & R^2 \\
\end{array}
\]

\[
\frac{Y_2}{X} = L^2 + 2L + 3 + 2R + R^2
\]
Example:

$$(\mathcal{L} + 1 + \mathcal{R}) \times (\mathcal{L} + 1 + \mathcal{R}) = \mathcal{L}^2 + 2\mathcal{L} + 3 + 2\mathcal{R} + \mathcal{R}^2$$
Cascading Systems

Generalize to non-causal DT systems.

\[
\frac{Y_1}{X} = \cdots + a_{-2}L^2 + a_{-1}L + a_0 + a_1R + a_2R^2 + \cdots
\]

\[
\frac{Y_2}{Y_1} = \cdots + b_{-2}L^2 + b_{-1}L + b_0 + b_1R + b_2R^2 + \cdots
\]
Cascading Systems

Generalize to non-causal DT systems.

\[
\frac{Y_2}{X} = \cdots + c_{-2}L^2 + c_{-1}L + c_0 + b_1R + c_2R^2 + \cdots 
\]

\[
\begin{array}{cccccc}
& \cdots & a_{-1}L & +a_0 & +a_1R & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_{-1}L & \cdots & a_{-1}b_{-1}L^2 & a_0b_{-1}L & a_1b_{-1} & \cdots \\
+b_0 & \cdots & a_{-1}b_0L & a_0b_0 & a_1b_0R & \cdots \\
+b_1R & \cdots & a_{-1}b_1 & a_0b_1R & a_1b_1R^2 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

\[
c_3 = \cdots + a_{-2}b_5 + a_{-1}b_4 + a_0b_3 + a_1b_2 + a_2b_1 \cdots = \sum_{k=-\infty}^{\infty} a_k b_{3-k}
\]

\[
c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k} \quad \text{Convolution Sum!}
\]
Cascading Systems

Cascade two systems \( \rightarrow \) convolve their impulse responses.

Let \( h_1[n] \) and \( h_2[n] \) represent the impulse responses of two systems in cascade:

\[
\begin{align*}
x[n] &\rightarrow h_1[n] & y_1[n] \\
h_1[n] &\rightarrow h_2[n] & y_2[n]
\end{align*}
\]

The impulse response of the cascade is then the convolution of \( h_1[n] \) and \( h_2[n] \):

\[
h[n] = (h_1 \ast h_2)[n] = \sum_{-\infty}^{\infty} h_1[k]h_2[n-k]
\]

Equivalent representation:

\[
x[n] \rightarrow (h_1 \ast h_2)[n] \rightarrow y_2[n]
\]
Signals as Systems

The response of a system with impulse response $h[n]$ is the convolution of its input signal $x[n]$ with $h[n]$.

Consider the cascade of a systems with impulse responses $x[n]$ and $h[n]$.

\[ \delta[n] \xrightarrow{x[n]} x[n] \rightarrow h[n] \rightarrow y[n] = (x \ast h)[n] \]

The impulse response of the cascade is the convolution of the impulse responses of the individual systems.

It follows that the response of the second system to $x[n]$ is the convolution of $x[n]$ with the impulse response of the second system.
Impulse Response

The impulse response is a new representation for systems. This representation is sometimes called the “time-domain” representation, since the signals and systems are both represented as functions of time.

This is in contrast to representing systems by system functions (e.g., functions of \( z \)) or functionals (e.g., \( \mathcal{L} \) and \( \mathcal{R} \)).
Convolution is closely associated with superposition.

All systems composed of delays, anticipators, scalers, and adders are “linear” and “time-invariant.”
**Linearity**

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given:

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]

The system is linear if the following input-output relation is true for all \( \alpha \) and \( \beta \).

\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]
**Time-Invariance**

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given:

\[ x[n] \xrightarrow{\text{system}} y[n] \]

The system is time invariant if the following input-output relation is true for all \( n_0 \).

\[ x[n - n_0] \xrightarrow{\text{system}} y[n - n_0] \]
If a system is linear and time-invariant (LTI) then its output can be determined using superposition.

$$x[n] = \sum_{n=-1}^{5} y[n]$$

where

$$y[n] = \sum_{n=-1}^{5} x[n]$$
Structure of Convolution

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \]
Structure of Convolution

\[ y[2] = \sum_{k=-\infty}^{\infty} x[k] h[2 - k] \]
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Structure of Convolution

\[ y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] \]
CT Convolution

Convolution of CT signals is completely analogous to convolution of DT signals.

\[
\text{DT: } y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]
\]

\[
\text{CT: } y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau
\]
**Impulse Response: Summary**

The impulse response is a complete description of a linear, time-invariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems.
Microscope

Images from even the best microscopes are blurred.
A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.

Blurring is inversely related to the diameter of the lens.
Microscope

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Microscope

A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.

Blurring is inversely related to the diameter of the lens.
We can characterize the blurring of a microscope by measuring its “impulse response.”

Image diameter $\approx 6$ times target diameter: target $\rightarrow$ impulse.
Images at different focal planes can be assembled to form a three-dimensional impulse response (point-spread function).
Blurring along the optical axis is better visualized by resampling the three-dimensional impulse response.

images by Anthony Patire
Blurring is much greater along the optical axis than it is across the optical axis.
The point-spread function (3D impulse response) is a useful way to characterize a microscope. It provides a direct measure of blurring, which is an important figure of merit for optics.
Hubble Space Telescope

Hubble Space Telescope (1990-)

http://hubblesite.org
Why build a space telescope?

Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.
Telescope blur can be represented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.

\[ h_a(\theta) \ast h_d(\theta) = h_t(\theta) \]

- \( h_a(\theta) \) is the atmospheric turbulence
- \( h_d(\theta) \) is the mirror size
- \( h_t(\theta) \) is the total blur

\( d = 12\text{cm} \)

\( d = 1\text{m} \)

[arc-seconds]
Hubble Space Telescope

The main optical components of the Hubble Space Telescope are two mirrors.

http://hubblesite.org
Hubble Space Telescope

The diameter of the primary mirror is 2.4 meters.

http://hubblesite.org
Hubble’s first pictures of distant stars (May 20, 1990) were more blurred than expected.

http://hubblesite.org
Hubble Space Telescope

The parabolic mirror was ground 4 µm too flat!

http://hubblesite.org
Corrective Optics Space Telescope Axial Replacement (COSTAR): eyeglasses for Hubble!
Hubble Space Telescope

Hubble images before and after COSTAR.

before

after

http://hubblesite.org
Hubble Space Telescope

Hubble images before and after COSTAR.

http://hubblesite.org
Hubble Space Telescope

Images from ground-based telescope and Hubble.

http://hubblesite.org
Impulse Response: Summary

The impulse response is a complete description of a linear, time-invariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.