1 Multiple representations

(a) Integrating twice gives the integral equation
\[ y(t) = \int_{-\infty}^{t} \int_{-\infty}^{t} x(t) dt \, dt + \int_{-\infty}^{t} \int_{-\infty}^{t} y(t) dt \, dt \] (1)
writing this in terms of functionals, we have
\[ \frac{Y}{X} = \frac{A^2}{1 - A^2} \] (2)
Substituting \( A = 1/s \) gives the transfer function \( H(s) \):
\[ H(s) = \frac{1}{s^2 - 1} = \frac{1}{(s + 1)(s - 1)} \] (3)
which has a pole at \(-1\), a pole at 1, and no zeroes. This is diagram A.
(b) This has a zero at 0, so by elimination, it is diagram F.
(c) It is non-oscillating, eliminating B, C, and E. It is stable, eliminating A. Since the step response decays to 0, it must also have at least one zero (poles could never by themselves cause a step response to decay). Thus, it is diagram F.
(d) Since the frequency response peaks, this is an oscillating high-Q system. Since it peaks to a finite height, it is not purely oscillating (infinite Q). This leaves choices B and C. Since \( Q = |\omega_0/2\alpha| = 2.87/1 \) it must be C by inspection, knowing the real and imaginary axes have the same scale. Alternatively, the ratio of the real part of the pole (|\alpha| to its radius \( \omega_0 \) is \( 1/2Q \), and knowing that \( Q = 2.87 \) it is easy to rule out B by inspection and pick C.
(e) Since there is a buffer, this circuit can be broken into two subsystems and the system functions of each can be multiplied together. Both are low-pass filters with system function \( 1/(sRC + 1) \) for their respective values of \( R \) and \( C \), so the first system has a pole at \(-1/(RC) = -1/0.69 \) sec\(^{-1} \approx -1.5 \) sec\(^{-1} \) and the second system has a pole at \(-1/(RC) = -1/10^{-1} \) sec\(^{-1} = -10 \) sec\(^{-1} \). There are no zeroes. This is diagram D.

2 Second-order systems

(a)
1. This system decays by a factor of \( 1/e \) in about 1.25 cycles, so \( Q/\pi = 1.25 \) as seen before in class. So \( Q \approx 1.25\pi \approx 4 \).
2. Since the impulse response does not begin with an impulse, and does not even begin with a step,
but is actually continuous, and is second-order (allowing two levels of integration), it must have no zeroes (zeroes would make it discontinuous again). With no zeroes, this is a lowpass filter, with a flat low-frequency magnitude asymptote.

(b)

1. Second order systems undergo a phase change by \( \pi \), which we see. They also undergo exactly half that phase change at \( \omega = \omega_0 \), so we can just read off that \( \omega_0 = 2 \cdot 10^3 \text{ rad/s} \).

2. One way to do this is to see that the phase starts at a flat \( 90^\circ \), which is characteristic of a differentiator \( j\omega = s \) and a single zero at 0. The phase ends at a flat \( -90^\circ \), which is characteristic of an integrator \( 1/(j\omega) = 1/s \), caused because the two poles of the second-order system changed the behavior of the system at around some frequency. Thus, if the magnitude \( |H(s)| \) looks like \( s \) for low frequencies and looks like \( s^{-1} \) for high frequencies, neither asymptote is flat. Alternatively, we also know that a second order system with only one zero is a bandpass (not lowpass or highpass) filter.

3. The bandwidth of the system is a little less than \( 2 \cdot 10^3 \text{ rad/s} \) (the approximate range over which the phase changes rapidly. Since \( \omega_0 = 2 \cdot 10^3 \), \( Q = \omega_0/\Delta \omega_0 \), so \( Q \) is a little higher than 1 by this estimate. The exact value for this particular data is \( Q = 1.5 \).

3 Block diagram

(a) The block diagram shows that \( Y_3 \) is obtained by multiplying \( Y_2 \) by 20 and integrating it. Thus, the two signals are related by the differential equation

\[
\frac{dy_3(t)}{dt} = 20y_2(t)
\]

(b) Each integrator smooths out the input by one order of discontinuity, so the second integrator before \( Y_3 \) will make \( Y_3 \) have a continuous impulse response and we don’t have to worry about the loop when dealing with values just after 0. In this case, \( Y_2 \) must look like 5 times a unit step just after \( t = 0 \) before the loop kicks in and changes this, but for \( t = 0^+ \), we can safely say \( y_2(0^+) = 5 \).

(c) Since we already decided that \( Y_3/X \) should have a continuous impulse response, \( y_3(0^+) = 0 \).

(d) We can write out the loop as \( 5 \cdot 20 \cdot A^2(X - Y_3) = Y_3 \). Solving for the functional,

\[
\frac{Y_3}{X} = \frac{100A^2}{1 + 100A^2}
\]

Converting this to a system function, say \( H_3(s) \),

\[
H_3(s) = \frac{100}{s^2 + 100}
\]

This has two poles at \( \pm 10j \) and no zeroes. This is the middle diagram in the bottom row.
4 Cascade

(a) The frequency response is given by $H(s)$ for $s = j\omega$. Plugging in the given value of $\omega = \sqrt{3}/\tau$ gives

$$H(j\omega) = \frac{1}{(1 + j\sqrt{3})^4} \quad (7)$$

Sketching $1 + j\sqrt{3}$ on the complex plane and using knowledge of the 30-60-90 triangle, $1 + j\sqrt{3}$ has a phase of $\pi/3$, so $(1 + j\sqrt{3})^4$ has a phase of $4\pi/3$, but this is in the denominator, so $H(j\omega)$ has a phase of $-4\pi/3$ (or $2\pi/3$).

(b) The closed-loop transfer function $H_c(s)$ is

$$H_c(s) = \frac{K H(s)}{1 + K H(s)} = \frac{K}{(1 + \tau s)^4 + K} \quad (8)$$

Solving the denominator for $s$ gives us the closed loop poles and zeroes:

$$(1 + \tau s)^4 + K = 0 \quad (9)$$

$$(1 + \tau s)^4 = -K = e^{j\pi} K \quad (10)$$

Using knowledge of vectors in the complex plane gives

$$(1 + \tau s) = e^{j\pi/4} K^{1/4}, \ e^{-j\pi/4} K^{1/4}, \ e^{3j\pi/4} K^{1/4}, \ e^{-3j\pi/4} K^{1/4} \quad (11)$$

$$\tau s = e^{j\pi/4} K^{1/4} - 1, \ e^{-j\pi/4} K^{1/4} - 1, \ e^{3j\pi/4} K^{1/4} - 1, \ e^{-3j\pi/4} K^{1/4} - 1 \quad (12)$$

The system will be unstable if any of the solutions on the right side are in the right half plane, that is, if their real parts are positive. Taking the real part of the entire expression and employing $\text{Re}[e^{j\phi}] = \cos(\phi)$ gives

$$\text{Re}[\tau s] = \cos(\pi/4) K^{1/4} - 1, \ \cos(-\pi/4) K^{1/4} - 1, \ \cos(3\pi/4) K^{1/4} - 1, \ \cos(-3\pi/4) K^{1/4} - 1 \quad (13)$$

$$\text{Re}[\tau s] = \pm \frac{1}{\sqrt{2}} K^{1/4} - 1 \quad (14)$$

This quantity will become positive (putting the pole in the right half plane) when $K^{1/4} > \sqrt{2}$, so it is first unstable when $K = 4$.

It is also possible to arrive at this result using the Nyquist criterion (discussed in the R15 notes): The phase of the open-loop system crosses $180^\circ$ when each pole contributes $45^\circ$, which happens at $\omega = 1/\tau$. At this frequency, each pole contributes $1/\sqrt{2}$ in magnitude, and since there are four poles, $|H(j\omega)| = 1/4$ at $\omega = 1/\tau$. So the total open-loop system magnitude $|K \cdot H|$ crosses 1 when $K = 4$. 

3
5 System analysis

(a) Inverting $H(s)$ gives a familiar form (with denominator $s^2 + \omega_0 s/Q + \omega_0^2$):

$$\frac{1}{H(s)} = \frac{s^2}{4 + s + 4s^2} = \frac{s^2}{s^2 + \frac{1}{4} s + 1}$$

which corresponds to a system with two zeroes at 0 (meaning the system begins with a slope of 2, and two poles (meaning the system ends with a slope of $2 - 2 = 0$). The two asymptotes intersect at $\omega = \omega_0 = 1$, and since $Q = 4$, it peaks to 4 times the height of the flat asymptote. The Bode plot of this looks like this:

To get the Bode plot of $H(s)$, we just have to invert this, which on log-log axes flips the graph around the horizontal line $|H(j\omega)| = 1$, giving us the final answer:

(b) This is best approached by taking the original system function and rewriting it as the sum of three terms:

$$H(s) = \frac{4 + s + 4s^2}{s^2} = \frac{4}{s^2} + \frac{s}{s^2} + \frac{4s^2}{s^2} = \frac{4}{s^2} + \frac{1}{s} + 4$$

(16)
Since $1/s$ corresponds to integration, we can think of this system as a weighted sum of a constant gain, an integrator, and a double integrator. For a step input $u(t)$, the output will then be $4u(t) + tu(t) + 4t^2u(t)$. Note that this output starts with a step at $t = 0$. The only choice that has this property is the left one in the second row.

The open-loop system has two poles at 0 and two zeroes at $-\frac{1}{8} \pm j\frac{3}{2}\sqrt{7}$, from the quadratic formula. The poles will split vertically, then move to the left and approach the zeroes as $K$ is increased.

![Root Locus](image)

6 Square roots forever

$$2 \times \sqrt{2} \times \sqrt{\sqrt{2}} \times \sqrt[4]{\sqrt{2}} \times \ldots = 2 \times 2^{1/2} \times 2^{1/4} \times 2^{1/8} \times \ldots = 2^{1/2 + 1/4 + 1/8 + \ldots}$$

(17)

Using the geometric series sum, this is just $2^{(1/(1-1/2))} = 2^2 = 4$.

Another way to approach this is to set the answer to some number, such as $P$:

$$2 \times \sqrt{2} \times \sqrt{\sqrt{2}} \times \sqrt[4]{\sqrt{2}} \times \ldots = P$$

(18)

This is equivalent to:

$$2 \times \sqrt[2]{2} \times \sqrt[4]{\sqrt{2}} \times \ldots = P$$

(19)

from which we can see that because of the infinite product, the left side is actually just $2\sqrt{P}$ itself:

$$2 \times \sqrt{P} = P$$

(20)

Solving this gives $\sqrt{P} = 2$ so $P = 4$. A similar technique (with addition of terms instead of products of terms) is one way to derive the infinite geometric sum formula directly.