Multiple Representations of DT Systems

A major focus of this subject has been the development of multiple representations of systems. Today we will spend some time thinking about relations among these representations.

\[ f(R) = \frac{1}{1 - R - R^2} \]

\[ H(z) = z^2 \]

\[ y[n] = x[n] + y[n-1] + y[n-2] \]

\[ \sum_{n=0}^{\infty} h[n] R^n \]

- Block diagram
- System functional
- Impulse response
- Series
- Partial fractions
- \( R \to \frac{1}{z} \)
- \( z \) transform

\[ z^2 - z - 1 \]

\[ 1, 1, 2, 3, 5, 8, ... \]
**Multiple Representations: Check Yourself**

Determine an expression for $H(z)$ in terms of $h[n]$.

**Multiple Representations of DT Systems**

In general, we should also include noncausal terms.

\[ \sum_{n=-\infty}^{1} h[n] \mathcal{L}^{-n} + h[0] + \sum_{n=1}^{\infty} h[n] R^n \]

Block diagram

System functional $f(R) = \frac{1}{1 - R - R^2}$

Impulse response $1, 1, 2, 3, 5, 8, \ldots$

Delay

 Partial fractions $\mathcal{R} \rightarrow \frac{1}{z}$

$L \rightarrow z$

System function $H(z) = \frac{z^2}{z^2 - z - 1}$

**Multiple Representations of DT Systems**

The $\mathcal{L}$ terms simply extend the sum to include negative values of $n$.

\[ f(R, \mathcal{L}) = \sum_{n=-\infty}^{1} h[n] \mathcal{L}^{-n} + h[0] + \sum_{n=1}^{\infty} h[n] \mathcal{L}^n \]

Replace $R$ with $\frac{1}{z}$ and $\mathcal{L}$ with $z$ to get

\[ H(z) = \sum_{n=-\infty}^{1} h[n] z^{-n} + h[0] + \sum_{n=1}^{\infty} h[n] z^{-n} \]

\[ H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} \]

This relation is called the **Z Transform**.
Z Transforms

Example: Find the Z transform of $h_1[n]$:

$$h_1[n] = \begin{cases} \left(\frac{7}{8}\right)^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We can represent $h_1[n]$ by the system functional

$$1 + \frac{7}{8}R + \left(\frac{7}{8}\right)^2 R^2 + \cdots = \frac{1}{1 - \frac{7}{8}R}$$

Substitute $R = z^{-1}$ to get

$$H_1(z) = \frac{1}{1 - \frac{7}{8}z^{-1}}$$

Z Transforms: Check Yourself

Find the Z transform of $h_2[n]$:

$$h_2[n] = \begin{cases} -\left(\frac{7}{8}\right)^n & \text{if } n < 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the relation between $H_2(z)$ and $H_1(z)$?

1. $H_2(z) = -H_1(z)$
2. $H_2(z) = H_1(z)$
3. $H_2(z) = \frac{1}{H_1(z)}$

Z Transforms

The Z transform $H(z)$ is completely specified by $h[n]$.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]$$

However, $h[n]$ is not completely specified by $H(z)$.

In the previous example, $H_1(z) = H_2(z)$ even though $h_1[n] \neq h_2[n]$. 
When we represent a system with its system function, we cannot tell whether the system is causal, anticausal, or non-causal.

Similarly, when we represent a signal by its Z transform, we cannot tell whether the signal is left-sided, right-sided, or both-sided.

Thus the Z transform is not a complete representation of a signal. We need additional information: the region of convergence (ROC).

Consider the Z transform of
\[ h[n] = \begin{cases} p^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \frac{1}{1-p/z} \]

But the sum only converges if \(|p/z| < 1\), i.e., \(|z| > |p|\).

Similarly, consider the Z transform of
\[ h[n] = \begin{cases} p^n, & \text{if } n \leq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{0} p^n z^{-n} = \sum_{l=-\infty}^{0} p^{-l}z^l = \frac{1}{1-z/p} \]

And the sum only converges if \(|z/p| < 1\), i.e., \(|z| < |p|\).
Z Transforms

Knowing the ROC of \( \frac{1}{1 - p z^{-1}} \) tells you whether it came from \( \frac{1}{1 - p R} \) or \( \frac{L}{L - p} \).

Z Transforms: Check Yourself

What is the ROC of the Z transform of the following function.

\[ h[n] = p^{|n|} \]

-4 -3 -2 -1 0 1 2 3 4

1. \(|z| < |p|
2. \(|z| < |1/p|
3. \(|p| < |z| < |1/p|
4. \(|1/p| < |z| < |p|
5. none of the above

Summary

Z transform + ROC: complete description.

\[ y[n] = x[n] + y[n-1] + y[n-2] \]

\[ h[n] = \sum_{n} h[n] L^{-n} + h[0] + \sum_{n} h[n] R^n \]

\[ f(R) = \frac{1}{1 - R - R^2} \]

\[ R \rightarrow \frac{1}{z} \]

\[ H(z) = \frac{z^2}{z^2 - z - 1} ; |z| > \phi \]
Multiple Representations of CT Systems

We can develop similar relations among the variety of representations that we have developed for CT systems.

Going from \( h(t) \) to \( f(A) \) is harder than going from \( h[n] \) to \( f(R, L) \).

Block diagram

System functional
\[
 f(A) = \frac{A}{1 - pA} \quad e^{st}u(t)
\]

Differential Equation
\[
 \frac{dy(t)}{dt} = x(t) + py(t)
\]

System function
\[
 H(s) = \frac{1}{s - p}
\]

Laplace transform

\[
 H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt
\]

Laplace Transforms

We can find the relation between \( H(s) \) and \( h(t) \) using the eigenfunction property and convolution.

Eigenfunction property:

If \( x(t) = e^{at} \) (for all time \( t \)), then \( y(t) = H(s)e^{at} \).

Calculate \( y(t) \) using convolution.

\[
 y(t) = H(s)e^{at} = (x \ast h)(t) = \int_{-\infty}^{\infty} e^{a(t-\tau)}h(\tau)d\tau = e^{at}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau
\]

Therefore

\[
 H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt
\]
**Z transforms**

You could have similarly derived the Z transform from the eigenfunction property and convolution. We didn’t because $h[n] \rightarrow f(\mathbb{R}, \mathcal{L})$ is so easy.

Eigenfunction property:

If $x[n] = z^n$ (for all time $n$), then $y[n] = H(z)z^n$.

Calculate $y[n]$ using convolution.

$$y[n] = H(z)z^n = (x \ast h)[n] = \sum_{k=-\infty}^{\infty} z^{n-k}h[k] = z^n \sum_{k=-\infty}^{\infty} z^{-k}h[k]$$

Therefore

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

**Laplace Transforms**

Example: Find the Laplace transform of $h_1(t)$:

$$h_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$h_1(t)$$

$$0$$

$$t$$

$$H_1(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} e^{-t}e^{-st}dt = \frac{e^{-(1+s)t}|_{0}^{\infty}}{-(1+s)} = \frac{1}{1+s}$$

provided $\text{Re}[1+s] > 0$ which implies that $\text{Re}[s] > -1$.

**Laplace Transforms: Check Yourself**

Find the function $h_2(t)$ whose Laplace transform is $H_2(s) = \frac{1}{1+s}$ with ROC $\text{Re}[s] < -1$.
**Laplace Transforms: Check Yourself**

Let $X(s)$ represent the Laplace transform of $x(t)$.

Determine the Laplace transform $Y(s)$ of $y(t) = x(t-T)$ where $T$ is a real-valued constant. Express $Y(s)$ in terms of $X(s)$.

**Z Transforms: Check Yourself**

Let $X(z)$ represent the Z transform of $x[n]$.

Determine the Z transform $Y(z)$ of $y[n] = x[n-n_0]$ where $n_0$ is an integer. Express $Y(z)$ in terms of $X(z)$.

**Summary**

Today: new relations among system representations.

- Block diagram
- System functional
- Impulse response
- Differential Equation
- System function + ROC
- Partial fractions
- Series

$dy(t)/dt = x(t) + py(t)$

$H(s) = \frac{1}{s - p}$; $\text{Re}[s] > p$