Lecture 17
Multiple Representations of Signals and Systems: Z and Laplace Transforms
Multiple Representations of DT Systems

A major focus of this subject has been the development of multiple representations of systems.

Today we will spend some time thinking about relations among these representations.
Multiple Representations of DT Systems

Block diagram

System functional

**Difference Equation**
\[ y[n] = x[n] + y[n-1] + y[n-2] \]

**System function**
\[ H(z) = \frac{z^2}{z^2 - z - 1} \]

**Impulse response**
\[ f(R) = \frac{1}{1 - R - R^2} \]

\[ \sum_0^\infty h[n]R^n \]

Series

Partial fractions

Z transform

1, 1, 2, 3, 5, 8, …
Multiple Representations: Check Yourself

Determine an expression for $H(z)$ in terms of $h[n]$. 
Determine an expression for $H(z)$ in terms of $h[n]$. 

It’s easy to write the system functional $f(R)$ in terms of the impulse response. It’s just

$$f(R) = \sum_{n=0}^{\infty} h[n]R^n.$$ 

Then convert the system functional $f(R)$ to the system function $H(z)$ by replacing $R$ with $\frac{1}{z}$.

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$
Multiple Representations of DT Systems

In general, we should also include noncausal terms.

\[ f(R) = \frac{1}{1 - R - R^2} \]

\[ H(z) = \frac{z^2}{z^2 - z - 1} \]

\[ \sum_{n=0}^{\infty} h[n]R^n + h[0] + \sum_{n=-\infty}^{1} h[n]L^n \]

Block diagram

System functional

Impulse response

1, 1, 2, 3, 5, 8, ...

Series

Partial fractions

\( R \rightarrow \frac{1}{z} \)

\( L \rightarrow z \)

Z transform
Multiple Representations of DT Systems

The $\mathcal{L}$ terms simply extend the sum to include negative values of $n$.

$$f(\mathcal{R}, \mathcal{L}) = \sum_{n=-\infty}^{-1} h[n] \mathcal{L}^{-n} + h[0] + \sum_{n=1}^{\infty} h[n] \mathcal{R}^{n}$$

Replace $\mathcal{R}$ with $\frac{1}{z}$ and $\mathcal{L}$ with $z$ to get

$$H(z) = \sum_{n=-\infty}^{-1} h[n] z^{-n} + h[0] + \sum_{n=1}^{\infty} h[n] z^{-n}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

This relation is called the **Z Transform**.
Z Transforms

Example: Find the Z transform of $h_1[n]$: 

$$h_1[n] = \begin{cases} 
\left(\frac{7}{8}\right)^n & \text{if } n \geq 0 \\
0 & \text{otherwise} 
\end{cases}$$

We can represent $h_1[n]$ by the system functional

$$1 + \frac{7}{8}R + \left(\frac{7}{8}\right)^2 R^2 + \cdots = \frac{1}{1 - \frac{7}{8}R}$$

Substitute $R = z^{-1}$ to get

$$H_1(z) = \frac{1}{1 - \frac{7}{8}z^{-1}}$$
Z Transforms: Check Yourself

Find the Z transform of $h_2[n]$:

$$h_2[n] = \begin{cases} -\left(\frac{8}{7}\right)^{-n} & \text{if } n < 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the relation between $H_2(z)$ and $H_1(z)$?

1. $H_2(z) = -H_1(z)$
2. $H_2(z) = H_1(z)$
3. $H_2(z) = \frac{1}{H_1(z)}$
Z Transforms: Check Yourself

Find the Z transform of \( h_2[n] \):

\[
h_2[n] = \begin{cases} 
-\left(\frac{8}{7}\right)^{-n} & \text{if } n < 0 \\
0 & \text{otherwise}
\end{cases}
\]

We can represent \( h_2[n] \) by the system functional

\[
-\left(\frac{8}{7}\right) L - \left(\frac{8}{7}\right)^2 L^2 - \left(\frac{8}{7}\right)^3 L^3 - \cdots = -\frac{8}{7} L \left( 1 + \left(\frac{8}{7}\right) L + \left(\frac{8}{7}\right)^2 L^2 + \cdots \right) = \frac{-\frac{8}{7} L}{1 - \frac{8}{7} L}
\]

Substitute \( L = z \) to get

\[
H_2(z) = \frac{-\frac{8}{7} z}{1 - \frac{8}{7} z} = \frac{1}{1 - \frac{7}{8} z^{-1}} = H_1(z)
\]
Z Transforms: Check Yourself

Find the Z transform of $h_2[n]$:

$$h_2[n] = \begin{cases} -\left(\frac{8}{7}\right)^{-n} & \text{if } n < 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the relation between $H_2(z)$ and $H_1(z)$?

1. $H_2(z) = -H_1(z)$
2. $H_2(z) = H_1(z)$
3. $H_2(z) = \frac{1}{H_1(z)}$
The Z transform $H(z)$ is completely specified by $h[n]$.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]$$

However, $h[n]$ is not completely specified by $H(z)$.

In the previous example, $H_1(z) = H_2(z)$ even though $h_1[n] \neq h_2[n]$. 
Z Transforms

When we represent a system with its system function, we cannot tell whether the system is causal, anticausal, or non-causal.

Similarly, when we represent a signal by its Z transform, we cannot tell whether the signal is left-sided, right-sided, or both-sided.

Thus the Z transform is not a complete representation of a signal. We need additional information: the region of convergence (ROC).
Z Transforms

Consider the Z transform of

\[ h[n] = \begin{cases} p^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \frac{1}{1 - p/z} \]

But the sum only converges if \(|p/z| < 1\), i.e., \(|z| > |p|\).
Similarly, consider the Z transform of

\[ h[n] = \begin{cases} p^n, & \text{if } n \leq 0 \\ 0, & \text{otherwise} \end{cases} \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{0} p^n z^{-n} = \sum_{l=-n}^{\infty} p^{-l}z^{l} = \frac{1}{1 - z/p} \]

And the sum only converges if \(|z/p| < 1\), i.e., \(|z| < |p|\).
Z Transforms

Knowing the ROC of \( \frac{1}{1 - p z^{-1}} \) tells you whether it came from \( \frac{1}{1 - p R} \) or \( \frac{L}{L - p} \).
What is the ROC of the Z transform of the following function.

\[ h[n] = p^{|n|} \]

- 1. \(|z| < |p|\)
- 2. \(|z| < |1/p|\)
- 3. \(|p| < |z| < |1/p|\)
- 4. \(|1/p| < |z| < |p|\)
- 5. none of the above
Find the region of convergence of the Z transform of the following function.

\[ h[n] = p^{|n|} \]

Break into two parts: one right-sided and one left-sided.

\[
h_R[n] = \begin{cases} p^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}
\]

\[
h_L[n] = \begin{cases} p^{-n}, & \text{if } n < 0 \\ 0, & \text{otherwise} \end{cases}
\]
Z Transforms

Find the Z transforms of $h_R[n]$ and $h_L[n]$.

$$H_R(z) = \sum_{n=\infty}^{\infty} h_R[n]z^{-n} = \sum_{n=0}^{\infty} p^nz^{-n} = \frac{1}{1-p/z} ; \quad |z| > |p|$$

This system has a pole at $z = p$.

$$H_L(z) = \sum_{n=\infty}^{\infty} h_L[n]z^{-n} = \sum_{n=-\infty}^{-1} p^{-n}z^{-n}$$

$$= \sum_{l=-n=1}^{\infty} p^lz^l = \frac{1}{1-pz} - 1 = \frac{pz}{1-pz} ; \quad |z| < |1/p|$$

This system has a pole at $z = 1/p$. 
The Z transform of the sum of $h_R[n]$ and $h_L[n]$ is the sum of the Z transforms:

$$H(z) = H_R(z) + H_L(z) = \frac{1}{1 - p/z} + \frac{pz}{1 - pz} = \frac{1 - p^2}{(1 - p/z)(1 - pz)}$$

provided $|p| < |z| < |1/p|$ (which also implies that $|p| < 1$).
What is the ROC of the Z transform of the following function.

\[ h[n] = |n|^p \]

1. \( |z| < |p| \)  
2. \( |z| < |1/p| \)  
3. \( |p| < |z| < |1/p| \)  
4. \( |1/p| < |z| < |p| \)  
5. none of the above
Summary

Z transform + ROC: complete description.

\[ f(\mathcal{R}) = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2} \]

\[ H(z) = \frac{z^2}{z^2 - z - 1} ; \quad |z| > \phi \]

System functional

Impulse response

\[ \sum_{n=-\infty}^{-1} h[n] \mathcal{L}^{-n} + h[0] + \sum_{1}^{\infty} h[n] \mathcal{R}^n \]

Series

Partial fractions

\[ R \rightarrow \frac{1}{z} \]

Z transform

\[ y[n] = x[n] + y[n-1] + y[n-2] \]

Difference Equation

Block diagram

System functional + ROC
Multiple Representations of CT Systems

We can develop similar relations among the variety of representations that we have developed for CT systems.
Multiple Representations of CT Systems

Going from \( h(t) \) to \( f(\mathcal{A}) \) is harder than going from \( h[n] \) to \( f(\mathcal{R}, \mathcal{L}) \).

\[
\frac{dy(t)}{dt} = x(t) + py(t)
\]

\[
H(s) = \frac{1}{s - p}
\]

Block diagram

System functional

\[
f(\mathcal{A}) = \frac{\mathcal{A}}{1 - p\mathcal{A}}
\]

Impulse response

\[
e^{pt}u(t)
\]

Laplace transform

\[
\mathcal{A} \rightarrow \frac{1}{s}
\]

Differential Equation System function

Series partial fractions
Laplace Transforms

We can find the relation between $H(s)$ and $h(t)$ using the eigenfunction property and convolution.

Eigenfunction property:

If $x(t) = e^{st}$ (for all time $t$), then $y(t) = H(s)e^{st}$.

Calculate $y(t)$ using convolution.

$$y(t) = H(s)e^{st} = (x \ast h)(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st} \int_{-\infty}^{\infty} e^{-s\tau}h(t)d\tau$$

Therefore

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$
Z transforms

You could have similarly derived the Z transform from the eigenfunction property and convolution. We didn’t because \( h[n] \to f(\mathcal{R}, \mathcal{L}) \) is so easy.

Eigenfunction property:

If \( x[n] = z^n \) (for all time \( n \)), then \( y[n] = H(z)z^n \).

Calculate \( y[n] \) using convolution.

\[
y[n] = H(z)z^n = (x \ast h)[n] = \sum_{k=-\infty}^{\infty} z^{n-k}h[k] = z^n \sum_{k=-\infty}^{\infty} z^{-k}h[k]
\]

Therefore

\[
H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}
\]
Laplace Transforms

Example: Find the Laplace transform of $h_1(t)$:

$$h_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_1(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} e^{-t}e^{-st}dt = \frac{e^{-(1+s)t}}{-(1+s)} \bigg|_{0}^{\infty} = \frac{1}{1+s}$$

provided $\text{Re}\{1+s\} > 0$ which implies that $\text{Re}\{s\} > -1$. 
Find the function $h_2(t)$ whose Laplace transform is

$$H_2(s) = \frac{1}{1 + s} \text{ with } \text{ROC } \text{Re}\{s\} < -1.$$
Laplace Transforms: Check Yourself

The pole at \( s = -1 \) means that the modes have the form \( e^{-t} \).

\[
H_2(s) = \int_{\alpha}^{\beta} Ce^{-t}e^{-st} \, dt = \frac{Ce^{-(1+s)t}}{-(1+s)} \bigg|_{\alpha}^{\beta}
\]

In order to converge for \( \text{Re}\{s\} < -1, \ t < 0 \). Try \( \alpha = -\infty, \beta = 0 \).

\[
H_2(s) = \int_{-\infty}^{0} Ce^{-t}e^{-st} \, dt = \frac{Ce^{-(1+s)t}}{-(1+s)} \bigg|_{-\infty}^{0} = \frac{-C}{1+s}
\]

Therefore \( C = -1 \).

\[
h_2(t)
\]

\[
h(t) = \begin{cases} 
-e^{-t} & \text{if } t \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]
Let $X(s)$ represent the Laplace transform of $x(t)$.

Determine the Laplace transform $Y(s)$ of $y(t) = x(t - T)$ where $T$ is a real-valued constant. Express $Y(s)$ in terms of $X(s)$.
Let $X(z)$ represent the Z transform of $x[n]$.

Determine the Z transform $Y(z)$ of $y[n] = x[n - n_0]$ where $n_0$ is an integer. Express $Y(z)$ in terms of $X(z)$. 

Summary

Today: new relations among system representations.

\[ f(A) = A - pA \]

\[ H(s) = \frac{1}{s - p}; \quad \text{Re}\{s\} > p \]

Block diagram

System functional

Impulse response

Differential Equation

\[ \frac{dy(t)}{dt} = x(t) + py(t) \]

Laplace transform

Series

Partial fractions