After studying this chapter, you should be able to:

- explain the analogy between Fourier coefficients and coordinates in a vector space;
- find the Fourier coefficients of a periodic signal; and
- analyze a system by how it alters Fourier coefficients.

We illustrate Fourier series and filters using a square-wave input signal fed into an RC circuit:

The problem is to find the output signal. It can be found numerically and perhaps even analytically by solving the leaky-tank or RC differential equation. Because the output signal is knowable without using Fourier series or filtering, feeding a square wave to an RC circuit is useful for practicing and understanding Fourier series and filtering.

In this chapter, we first find the Fourier representation of the square wave. Then we can compute the output signal in two ways – using the time representation or the Fourier representation – and can compare the results. Rather than exactly solving differential equations or calculating general and therefore messy Fourier sums, we’ll analyze the system in the two extreme cases of a fast and a slow square wave.

18.1 Fourier representation of signals

In the Fourier representation, the square wave is written as a weighted sum of oscillating exponentials (or of sines and cosines). In this representation, the circuit filters the signal: It adjusts the weights of the exponentials to produce the Fourier representation of the output signal. A system’s operation as a filter is often simpler to understand than its operation as a differential equation.

The Fourier representation is

\[
\text{square wave} = \sum_{k=-\infty}^{\infty} a_k F_k
\]

where \(a_k\) is the \(k\)th Fourier coefficient or weight; \(F_k\) is the \(k\)th Fourier basis function, given by

\[
F_k \equiv e^{j\omega_k t},
\]

and \(\omega_k \equiv 2\pi k/T\) is the angular frequency of the \(k\)th Fourier function. The Fourier weights \(a_k\) that represent a function \(f(t)\) are given by the Fourier inversion formula:
\[ a_k = \frac{1}{T} \int_T f(t)e^{-j\omega_k t} \, dt, \]

where \( T \) is the period of the function \( f(t) \) and, as a subscript on the integral, indicates integration over one period.

### 18.1.1 Function space

To understand these formulas, think of the Fourier representation as a coordinate system in function space, which is a kind of vector space. In a finite-dimensional vector space, for example the Euclidean plane, any vector \( r \) is a weighed sum of the unit vectors \( \hat{x} \) and \( \hat{y} \) along the coordinate axes. In that form,

\[ r = a\hat{x} + b\hat{y}, \]

where \( a \) and \( b \) are the coordinates or weights. In function space, the coordinate axes of this space are the functions \( F_k \), and the coordinates of \( f(t) \) are its Fourier weights \( a_k \). Then the Fourier-representation formula

\[ f(t) = \sum_{k=-\infty}^{\infty} a_k F_k \]

is an infinite-dimensional version of the familiar two-dimensional form.

How do you compute the weights \( a_k \)? In a finite-dimensional vector space with perpendicular axes, you find each coordinate of a vector by computing its dot product with a unit vector along that axis. Similarly, to find the coordinates of a function \( f(t) \) in this function space, take the dot product of \( f(t) \) with each basis vector \( F_k \). The dot product of two vectors \( b = (b_0, b_1, b_2, \ldots) \) and \( c = (c_0, c_1, c_2, \ldots) \) is the sum of componentwise products:

\[ b \cdot c = \sum b_k c_k. \]

From here we reach the Fourier inversion formula in three steps. The first step is to account for complex values. In a space with imaginary (or complex) basis vectors, we take the complex conjugate of the second vector, making the dot product

\[ b \cdot c = \sum b_k c_k^*. \]

This change ensures that the dot product of a vector with itself, which should be a squared length, is real and positive. The second step is to account for the infinite dimensionality. In an infinite-dimensional space, the sum becomes an integral over time. The third step is to account for the lengths of the vectors. We would like the basis functions \( F_k \) to be unit vectors, i.e. to have unit length. So we would like \( F_k \cdot F_k \) to be 1. So we put a factor of \( 1/T \) in front of the integral and define the infinite-dimensional dot product as

\[ f \cdot F_k \equiv \frac{1}{T} \int_T f(t)F_k^*(t) \, dt. \]

The basis functions are \( F_k = e^{j\omega_k t} \), so the Fourier inversion formula
\[ a_k = \frac{1}{T} \int_T f(t) e^{-j\omega_k t} \, dt, \]
says that
\[ a_k = f \cdot F_k, \]
which is what you would find in a finite-dimensional space. [The minus sign in the exponent comes from the complex conjugation.]

### 18.1.2 Computing the coordinates (weights)

To compute the weight integral for the square wave, we need to choose where to put zero time and zero voltage. Since the circuit is time invariant, it does not matter where we put the origin of time. A choice that simplifies subsequent integrals is to take one of the pulses in the square wave and call its leading edge the time origin. It also does not matter where we put zero voltage, because we can measure all input and output voltages relative to that reference level. Use freedom to increase symmetry, and choose the most symmetric location for zero voltage, which is to place it halfway between the low and high levels of the square wave, and say that the high level is \( V = 1/2 \).

![Image of a square wave](image)

**Exercise 80.** The full argument about zero voltage is more subtle. Why, for this system, does it not matter where we place the zero voltage level?

It is also convenient to choose the integration range. Since \( f \) and \( e^{-j\omega_k t} \) are periodic with period \( T \), the integrand in their dot product is also periodic with period \( T \). Therefore we can integrate over any convenient interval of length \( T \). Use freedom to increase symmetry. For the square wave, integrating from \(-T/2\) to \( T/2\) is the most symmetric choice, and probably the least messy:

\[ a_k = \frac{1}{2T} \left( \int_{-T/2}^{T/2} - \int_{-T/2}^{0} \right) e^{-j\omega_k t} \, dt, \]
since \( f(t) = 1/2 \) from \( t = 0 \) to \( T/2 \), and \( f(t) = -1/2 \) from \( t = -T/2 \) to \( 0 \). Now do the integral:

\[ a_k = -\frac{1}{j2T \omega_k} \left( (e^{-j\omega_k T/2} - 1) - (1 - e^{j\omega_k T/2}) \right). \]

Since \( \omega_k T = 2\pi k \), this mess simplifies to

\[ a_k = \frac{1}{j2\pi k} \left( 1 - e^{-j\pi k} \right) = \begin{cases} \frac{1}{j\pi k} & (k \text{ odd}) \\ 0 & (\text{otherwise}) \end{cases}. \]

The Fourier representation is

\[ \text{square wave} = \sum_{\text{odd } k} \frac{e^{j\omega_k t}}{j\pi k}. \]

This sum looks complex (in the sense of having an imaginary part). However, the square wave is real. So the coefficients conspire to make the sum real. To see how, pair up corresponding terms.
For example, the $k = 5$ term is $a_5 e^{j\omega t}$. Its partner is the $k = -5$ term, which is $a_{-5} e^{j\omega t}$. Since $a_{-5}$ is the complex conjugate of $a_5$, and $e^{j\omega t}$ is the complex conjugate of $e^{-j\omega t}$, the product $a_{-5} e^{j\omega t}$ is the complex conjugate of $a_5 e^{j\omega t}$. So

$$a_{-5} e^{j\omega t} + a_5 e^{j\omega t} = 2 \text{Re} a_5 e^{j\omega t}.$$  

With $a_k = 1/(j\pi k)$, the sum of the paired terms is, for general $k$:

$$\frac{2\sin(2\pi k t/T)}{\pi k}.$$  

So

$$\text{square wave} = \sum_{k \text{ odd, positive}}^{\infty} \frac{2\sin(2\pi k t/T)}{\pi k}.$$  

Should the Fourier representation contain only sines? To decide, think about symmetry. With our choice of origin, the square wave is antisymmetric, meaning that $f(t) = -f(-t)$. So it should be composed only out of antisymmetric functions. Since sines are antisymmetric and cosines are symmetric, the square wave should indeed contain only sines.

Now that we have the Fourier representation for the square wave, we have a new method to find the output signal: See what the circuit does to each Fourier component, and figure out what signal has that representation. We do so for the two extremes of the input: a fast and a slow square wave.

### 18.2 Slow square wave

The slow square wave is easier to analyze than is the fast square wave, so analyze the slow case first. ‘Slow’ has only a relative meaning: slow compared to another time. The only other time in this problem is the time constant $\tau$ of the RC circuit. So if the period $T$ is long compared to $\tau$, then the square wave is slow. We use the time representation to reason out the response to a slow square wave, then confirm the result with the Fourier representation.

#### 18.2.1 Time representation

A true square wave starts at $t = -\infty$. But to start the time-representation analysis, instead imagine that the input signal is 0 for $t < 0$, and then turn on the square-wave signal at $t = 0$. Eventually the transient produced by turning on the input at $t = 0$ will settle down, but even before then, we can understand the main features of the output.

The first segment of the square-wave input is a positive voltage until $t = T/2$. So the RC circuit first sees a positive step that lasts for a time $T/2$. In the slow-square-wave extreme, the period $T$ is much larger than $\tau$. So as far as the RC circuit is concerned, the first segment of the square wave lasts forever, and the capacitor has a lot of time to charge to its steady-state value of $V = 1/2$. The output signal is an exponential approach to $1/2$:

![Graph showing exponential approach to V = 1/2](image)
After many \( RC \) time constants, the second segment of the square wave arrives. Relative to the first segment, it is a negative step that discharges the capacitor toward \(-1/2\). The output voltage gets most of the way to \(-1/2\) in a few \( RC \) time constants. Eventually the third segment of the square wave arrives and the \( RC \) adjusts to the new voltage level long before the fourth segment arrives.

Except for a short interval after each transition, the output voltage tracks the input voltage almost exactly, meaning that the \( RC \) circuit acts like a wire. Let’s see if we can understand that result using Fourier analysis.

### 18.2.2 Fourier representation of filtering

In the Fourier representation, the operation of the \( RC \) circuit is particularly simple. The square wave is a weighted sum of complex exponentials (or of sines), and the \( RC \) circuit merely adjusts the weights. The Fourier coefficient \( a'_k \) of the output signal is

\[
a'_k = a_k H(j\omega).
\]

To picture this product, the Bode plot for an \( RC \) circuit is helpful because its logarithmic magnitude axis converts multiplication into addition. So, on the Bode magnitude plot overlay the Fourier-coefficient magnitudes \(|a_k|\), associating each one with its frequency \( \omega_k = 2\pi k/T \). Here is a picture with \( T = 100\tau \) showing the first several coefficients as dots:

![Bode plot of Fourier representation of filtering](figure)

The first dots lie in the frequency region of the unity-gain asymptote, so their magnitudes are unscathed by the \( RC \) circuit. Their phase is also untouched as long as their frequency is much less than \( 1/\tau \), because the low-frequency phase asymptote is \( 0° \).

The higher-frequency Fourier components lie in the frequency region of the downward-sloping magnitude asymptote. So the \( RC \) circuit shrinks the high-frequency components. However, their amplitudes started small compared to the amplitudes of the low-frequency components, and the \( RC \) circuit makes these small amplitudes even tinier. These small changes affect the signal but not greatly, so the output signal is almost a replica of the square wave. However, the sharp edges are not replicated. The reason is that discontinuities require high (actually infinite) frequency, and the filtering squashes those high frequencies, so the sharp edges become smooth. These features are reflected in the output signal derived using the time representation.

### 18.3 Fast square wave

The other extreme of the input signal is a fast square wave whose period \( T \) is much smaller than the \( RC \) time constant \( \tau \). We will use the Fourier representation to predict the output, and leave you to confirm the result using the time representation (for example, by solving differential equations).
18.3.1 Fourier representation of filtering

A fast square wave contains the frequencies \( \omega_k = 2\pi k/T \) (for odd \( k \)) but now the period \( T \) is very short, which means means \( T \ll \tau \). So a fast square wave, reasonably enough, contains high frequencies. Here is the RC Bode magnitude picture overlaid with the Fourier amplitudes:

\[
\begin{array}{c}
\omega = 1/\tau \\
2\pi/T \\
\hline
1 \\
3 \\
5 \\
\vdots
\end{array}
\]

The dots look as they did for the slow square wave except that they are shifted to the right. This pictorial invariance is one reason to use a logarithmic frequency scale. The component frequencies are in the ratio \( 1 : 3 : 5 : \cdots \), no matter the period. The period determines the fundamental frequency (labelled with a ‘1’), but not those ratios. On a logarithmic scale, those ratios turn into relative distances, so the relative distances – the pattern – is independent of period. Changing the period just shifts the pattern.

In their new position, the Fourier coefficients (the dots) are significantly altered by the RC circuit. The square-wave frequencies live in the frequency region of the downward-sloping asymptote rather than of the flat asymptote. The downward-sloping asymptote has a \(-1\) slope, so the RC filter contributes a magnitude proportional to \( 1/\omega \). Since \( \omega \sim k \), where \( k \) is the index of the Fourier basis function \( F_k \), the RC filter contributes a factor proportional to \( 1/k \). So

\[
|a'_k| \sim \frac{1}{\pi k} \times \frac{1}{k} \sim \frac{1}{k^2} \quad \text{(odd } k\text{)}.
\]

To find the output signal from the Fourier coefficients, we also need to know the phase of \( a'_k \). For a sufficiently fast square wave, all the dots lie in the region of the \(-90^\circ\) phase asymptote, which begins at \( \omega = 10/\tau \). A \(-90^\circ\) phase means a factor of \( 1/j \). So the output Fourier coefficients pick up a factor of \( 1/j \), giving

\[
a'_k \sim \frac{1}{j\pi k} \times \frac{1}{k} \times \frac{1}{j} = -\frac{1}{\pi k^2} \quad \text{(odd } k\text{)}.
\]

To find the resulting output signal we have to go the other direction: from the Fourier coefficients \( a_k \sim 1/k^2 \) to the function of time with those coefficients. That problem is hard in general, but in this case the solution is given in lecture. The \( 1/k^2 \) coefficients (for odd \( k \)) produce a triangle wave:
Exercise 81. We were incomplete when we said that the downward-sloping asymptote contributed a magnitude factor proportional to $1/k$. Include the missing constant of proportionality to get the output signal’s Fourier coefficients as a function of $T$ (in the $T \ll \tau$ limit). Then find the amplitude of the resulting triangle wave as a function of $T$.

Exercise 82. Confirm the result of this section – that the output is a triangle wave – by analyzing the circuit in the time representation (again for $T \ll \tau$), perhaps by approximating the differential equation in that limit or by thinking about how the circuit behaves for short times.

18.4 Summary

Now you know another representation of signals: the Fourier representation. And you know another representation of systems: as filters that alter Fourier coefficients.

Exercise 83. What happens in the intermediate case, where the period $T$ and the time constant $\tau$ are comparable? In particular, what happens in the case where the first few dots lie in the region of the unity-gain asymptote but in the region of the sloping phase asymptote?