Lecture 19

Multiple Representations of Signals:
CT Fourier Transforms
Multiple Representations of Signals

Last time, we developed a new representation for signals: **Fourier Series**, in which signals are constructed from sums of sinusoids.

That new representation for signals led to a new representation for systems as **filters**.

Today, we will generalize those results for periodic signals to aperiodic signals.
Fourier Transforms

A signal that is not periodic can be thought of as being periodic with an infinite period.

Let \( x(t) \) represent an aperiodic signal.

\[
\begin{align*}
x(t) & = \lim_{T \to \infty} x_T(t).
\end{align*}
\]

"Periodic extension": \( x_T(t) = \sum_{k=-\infty}^{\infty} x(t + kT) \)
Fourier Transforms

Represent $x_T(t)$ by its Fourier series.

$$x_T(t)$$

$-S \quad S \quad T$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt} \, dt = \frac{1}{T} \int_{-S}^{S} e^{-j\frac{2\pi}{T}kt} \, dt = \frac{\sin \frac{2\pi kS}{T}}{\pi k} = \frac{2}{T} \frac{\sin \omega S}{\omega}$$

$$T a_k \quad \frac{2 \sin \omega S}{\omega} \quad \omega = k \omega_0 = k \frac{2\pi}{T}$$

$$\omega_0 = \frac{2\pi}{T}$$
Doubling the period doubles the number of harmonics in a given frequency interval.

\[ a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-j \frac{2\pi}{T} kt} \, dt = \frac{1}{T} \int_{-S}^{S} e^{-j \frac{2\pi}{T} kt} \, dt = \frac{\sin \frac{2\pi kS}{T}}{\pi k} = \frac{2}{T} \frac{\sin \omega S}{\omega} \]

\[ T a_k \]

\[ \omega = k \omega_0 = k \frac{2\pi}{T} \]

\[ \omega_0 = 2\pi / T \]
Fourier Transforms

As the period goes to infinity, the amplitudes of the harmonics converge to a function of $\omega$.

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \int_{-S}^{S} e^{-j\frac{2\pi}{T}kt} dt = \frac{\sin \frac{2\pi kS}{T}}{\pi k} = \frac{2}{T} \frac{\sin \omega S}{\omega}$$

$$\omega = k \omega_0 = k \frac{2\pi}{T}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\lim_{T \to \infty} T a_k = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \frac{2}{\omega} \sin \omega S = E(\omega)$$
In the limit $T \to \infty$, the Fourier series representation of $x(t)$ as a sum of harmonics passes to an integral of the function $E(\omega)$.

$$x_T(t)$$

$$\lim_{T \to \infty} T a_k = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt = \frac{2}{\omega} \sin \omega S = E(\omega)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} E(\omega) e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} E(\omega) e^{j\omega t} \to \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega) e^{j\omega t} \, d\omega$$
Fourier Transforms

The function $E(\omega)$ is the Laplace transform of $x(t)$ evaluated at $s = j\omega$. Replacing $E(\omega)$ by $X(j\omega)$ yields the Fourier transform relations.

Fourier transform

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \text{("analysis" equation)}
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \text{("synthesis" equation)}
\]
Relation between Fourier and Laplace Transforms

If the Laplace transform of a signal exists and if the ROC includes the $j\omega$ axis, then the Fourier transform is equal to the Laplace transform evaluated on the $j\omega$ axis.

Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Fourier transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = H(s)|_{s=j\omega}$$
**Relation between Fourier and Laplace Transforms**

Compare Fourier and Laplace transforms of $x(t) = e^{-t}u(t)$.

**Laplace transform**

$$X(s) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+1)t}\,dt = \frac{1}{1+s}; \quad \text{Re}\{s\} > -1$$

**Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(j\omega+1)t}\,dt = \frac{1}{1+j\omega}$$
Laplace Transforms

The Laplace transform maps a function of time $t$ to a complex-valued function of complex-valued domain $s$.

$$|X(s)| = \left| \frac{1}{1 + s} \right|$$
Fourier Transforms

The Fourier transform maps a function of time $t$ to a complex-valued function of real-valued domain $\omega$.

$$|X(j\omega)| = \left|\frac{1}{1 + j\omega}\right|$$
Fourier Transforms

A function of real domain $\omega$, the Fourier transform is often easier to visualize than the equivalent Laplace transform.

Example: square pulse

$$x_1(t)$$

$\begin{array}{c}
\text{1} \\
\text{1}
\end{array}$

Laplace transform:

$$X(s) = \int_{-1}^{1} e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_{-1}^{1} = \frac{1}{s} (e^s - e^{-s}) \quad \text{[function of } s = \sigma + j\omega]\]$$

Fourier transform

$$X(j\omega) = \int_{-1}^{1} e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-1}^{1} = \frac{2 \sin \omega}{\omega} \quad \text{[function of } \omega]\]$$
The magnitude of this Laplace transform grows exponentially as $\text{Re}\{s\}$ increases or decreases.

\[ |X_1(s)| = \left| \frac{1}{s} (e^s - e^{-s}) \right| \]
**Fourier Transform**

The Fourier transform is easier to visualize. It is a function of a single variable: frequency $\omega$.

**Time representation:**

$$x_1(t)$$

![Time representation diagram](image)

**Frequency representation:**

$$X_1(j\omega) = \frac{2\sin \omega}{\omega}$$

![Frequency representation diagram](image)
The Fourier transform of $x_2(t)$ is $X_2(j\omega)$ shown below.

Which of the following is true?

1. $b = 1$ and $\omega_0 = \pi/2$
2. $b = 1$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above
Fourier Transforms: Check Yourself

We could just substitute $x_2(t)$ into the definition of the Laplace transform. Instead, let’s find a general scaling rule.

Let $x_2(t) = x_1(at)$.

For our problem, is $a$ equal to $1/2$ or $2$?
We could just substitute $x_2(t)$ into the definition of the Laplace transform. Instead, let’s find a general scaling rule.

Let $x_2(t) = x_1(at)$.

For our problem, is $a$ equal to 1/2 or 2?

$$x_2(2) = x_1(1)$$

$$x_2(t) = x_1(at)$$

Therefore $a = 1/2$. 
Fourier Transforms: Check Yourself

We could just substitute $x_2(t)$ into the definition of the Laplace transform. Instead, let’s find a general scaling rule.

Let $x_2(t) = x_1(at)$.

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt = \int_{-\infty}^{\infty} x_1(at)e^{-st}dt$$

Let $\tau = at$.

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau/a}d\tau/a = \frac{1}{a}X_1(j\omega/a)$$

Here we have assumed $a > 0$. If $a < 0$ then $dt = d\tau/|a|$.

$$x_1(at) \leftrightarrow \frac{1}{|a|}X_1(j\omega/a)$$

$$a = \frac{1}{2} : \quad b = 2 \times 2 = 4 ; \quad \omega_0 = \frac{\pi}{2}$$
The Fourier transform of $x_2(t)$ is $X_2(j\omega)$ shown below.

Which of the following is true?

1. $b = 1$ and $\omega_0 = \pi/2$
2. $b = 1$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above
Fourier Transforms

Stretching time compresses frequency.

\[ x_1(t) \]

\[ X_1(j\omega) = \frac{2\sin \omega}{\omega} \]

\[ x_2(t) \]

\[ X_2(j\omega) = \frac{4\sin 2\omega}{2\omega} \]
Fourier Transforms

One of the most useful features of the Fourier transform (and Fourier series) is the simple “inverse” Fourier transform.

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]  
\text{(Fourier transform)}

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]  
\text{ ("inverse" Fourier transform)}

The Fourier transform and its inverse have very similar forms.

Convert one to the other by

- \( t \rightarrow \omega \)
- \( \omega \rightarrow -t \)
- scale by \( 2\pi \)
Find the impulse response of an “ideal” low pass filter.
Fourier Transforms: Check Yourself

\[ H(j\omega) = \begin{cases} 1 & -\omega_0 < \omega < \omega_0 \\ 0 & \text{otherwise} \end{cases} \]

\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega = \frac{1}{2\pi} \left. e^{j\omega t} \right|_{-\omega_0}^{\omega_0} = \frac{\sin \omega_0 t}{\pi t} \]

This result is not so easily obtained without using the inverse Fourier transform relation.
Filtering

Representing signals by their Fourier transforms allows us to think about systems by the way they “filter” signals based on their frequency content.
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Representing signals by their Fourier transforms allows us to think about systems by the way they “filter” signals based on their frequency content.

The Fourier transform represents a signal as a sum of complex exponentials.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \]

Complex exponentials are eigenfunctions of LTI systems.

\[ e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t} \]

The system “filters” the input by adjusting the amplitude and phase of each frequency component.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t} d\omega \]
Filtering

Systems can be designed to selectively pass certain frequency bands. Examples: low-pass filter (LPF) and high-pass filter (HPF).
Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.

ECG and analysis by T. F. Weiss
Filtering Example: Electrocardiogram

In addition to picking up electrical responses of the heart, electrodes on the skin also pick up a variety of other electrical signals that we regard as “noise.”

We wish to design a filter to eliminate the noise.

\[ x(t) \rightarrow \text{filter} \rightarrow y(t) \]
Filtering Example: Electrocardiogram

We can identify the “noise” by breaking the electrocardiogram into frequency components using the Fourier transform.

\[ f = \frac{\omega}{2\pi} \text{ [Hz]} \]
Filtering Example: Electrocardiogram

Filter design: low-pass filter + high-pass filter + notch.
Electrocardiogram: Check Yourself

Which poles and zeros are associated with
- the high-pass filter?
- the low-pass filter?
- the notch filter?
Electrocardiogram: Check Yourself

Which poles and zeros are associated with
- the high-pass filter?
- the low-pass filter?
- the notch filter?
Filtering Example: Electrocardiogram

By placing the poles of the notch filter very close to the zeros, the width of the notch can be made quite small.

\[ f = \frac{\omega}{2\pi} \text{ [Hz]} \]
Filtering Example: Electrocardiogram

Comparison of filtered and unfiltered electrocardiograms.

\[ x(t) \text{ [mV]} \]

\[ y(t) \text{ [mV]} \]

\[ f = \frac{\omega}{2\pi} \text{ [Hz]} \]
Filtering Example: Electrocardiogram

Reducing the frequency components that are not generated by the heart simplifies the output, making it easier to diagnose cardiac problems.

Unfiltered ECG

![Unfiltered ECG graph]

Filtered ECG

![Filtered ECG graph]
Continuous-Time Fourier Transforms: Summary

Fourier transforms represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., electrocardiogram.

Representing a signal by its frequency content motivates representing a system as a filter.

Representing a system as a filter is useful for many systems, e.g., electrocardiogram.