6.003 (Fall 2007): Homework 11

Do all of the following problems, including the question about hours spent on the problem set. Due in recitation on Wednesday, 21 November 2007.

Warmup problems

1. Practice with transforms
   Find the Fourier transforms of these functions of time. For each one, sketch $|f(\omega)|$.
   
   a. $\delta(t)$
   b. 1
   c. $\sin t$
   d. $\cos t$
   e. $1 + \cos 2t$
   f. $\cos^2 t$
   g. $e^{-t}$ (for $t \geq 0$)

   It is often easier to try $f(\omega)$ candidates until you find one whose inverse transform is the desired $f(t)$.

2. Practice with inverse transforms
   Find and sketch the inverse Fourier transforms of these functions of frequency.
   
   a. $\delta(\omega)
   b. \delta(\omega - 2) + \delta(\omega + 2)$
   c. 1

3. Multiplication of areas
   a. Use Fourier transforms to show that
   \[
   \text{Area of } \{f \ast g\} = \text{Area of } f \times \text{Area of } g.
   \]
   
   b. Sketch the convolution of this pulse with itself and confirm the area property in part (a):

4. Periodic signal
   Find the Fourier transform of the square wave analyzed in Lecture 18:

\[
\begin{align*}
\text{Area of } \{f \ast g\} = \text{Area of } f \times \text{Area of } g.
\end{align*}
\]
5. Discrete sinusoids
Consider two DT signals:
\[ x_1[n] = \cos\left(\frac{3\pi n}{4}\right) \quad \text{and} \quad x_2[n] = \cos\left(\frac{5\pi n}{4}\right). \]

How many of the following statements are true?
1. \(x_1[n]\) is periodic with a period of 8.
2. \(x_2[n]\) is periodic with a period of 8.
3. \(x_1[n] = x_2[n]\).
Briefly explain.

Medium problems

6. Delay
a. What is the Fourier transform of \(Rf\) in terms of \(f(\omega)\), the Fourier transform of \(f(t)\)? The \(R\) operator is our old friend that right shifts by one time unit (and it works for CT as well as DT signals).

b. Sketch the inverse transform of \(f(\omega) = \frac{e^{10j\omega}}{1 - j\omega}\).

c. Find the transform of this pulse:

\[
\begin{array}{c|c|c}
0 & 1.5 & 4 \\
\hline
0 & & 4
\end{array}
\]

Hint: The Lecture 19 notes give the transform of a similar signal.

7. Two-sided exponential decay
a. Sketch the function \(f(t) = e^{-|t|},\) where \(t\) ranges from \(-\infty\) to \(\infty\).

b. Show that its Fourier transform is
\[ f(\omega) = \frac{2}{1 + \omega^2} \]
and sketch \(f(\omega)\).
c. Find the system function $H(s)$ for the system with $f(t)$ as its impulse response. Mark its poles on a pole–zero diagram, and give the region of convergence.

d. Without doing any more integrals, sketch the Fourier transform of $e^{-|t|/2}$, giving scales on the $\omega$ and $f(\omega)$ axes.

**Harder problems**

8. Filtered impulse train

You send a periodic train of delta functions (with unit period) into an RC filter with $\tau = 1$.

a. What is the DC value (i.e. the average value) of the output signal?

b. Sketch the output signal, and mark the DC value as a horizontal line.

c. What is the peak value of the output signal?

Feel free to simulate, to build a circuit, to work out the quantities analytically, . . . . Ideally you would use more than one method and check that they agree.

9. Smoothing by convolution

The function $f(t) = e^{-|t|}$ has a discontinuous slope at $t = 0$, which is a rarity in physical systems. To smooth the discontinuity, one can convolve $f$ with itself. The resulting function is a reasonable approximation to the impulse response of a lens, which is why we are asking you to study it in this question.

a. Find and sketch $f \ast f$.

b. Find the function $f(t)$ whose Fourier transform is

$$f(\omega) = \left( \frac{2}{1 + \omega^2} \right)^2$$

In other words, find the inverse Fourier transform of $f(\omega)$.

There are at least three methods, in order of increasing difficulty: convolution, partial fractions, and contour integration of the inversion formula. Use your two favorite methods and confirm that they give the same $f(t)$. If you use partial fractions, you might find the Recitation 11 notes useful for tips on maintaining algebra hygiene.

10. Hours

While our primary goal in designing homework assignments is that these exercises should be educational, we know that they take time. Please help us determine how reasonable the workload in 6.003 is by estimating how many hours you spent during the past week working on this homework assignment. Feel free also to comment on these problems.