Lecture 21
Relations among Fourier Representations
Different Fourier representations are related because they apply to signals that are related.

- **DTFS (discrete-time Fourier series):** periodic DT
- **DTFT (discrete-time Fourier transform):** aperiodic DT
- **CTFS (continuous-time Fourier series):** periodic CT
- **CTFT (continuous-time Fourier transform):** aperiodic CT
Duality of the Fourier transform

Because the forward and inverse Fourier transform relations are so similar, every transform pair has a dual.

If \( x(t) \leftrightarrow X(j\omega) = g(\omega) \)

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega)e^{j\omega t} d\omega
\]

\[
g(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt
\]

Let \( \omega \rightarrow -t, \ t \rightarrow \omega \), and multiply and divide by \( 2\pi \):

\[
g(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi x(\omega)e^{j\omega t} d\omega
\]

then \( g(-t) \leftrightarrow 2\pi x(\omega) \)
Duality of the Fourier transform

\[ x(t) \leftrightarrow g(\omega) \]
\[ \omega \rightarrow -t \quad t \rightarrow \omega; \times 2\pi \]
\[ g(-t) \leftrightarrow 2\pi x(\omega) \]

\[ x(t) = \delta(t) \]
\[ t \quad 1 \]
\[ \omega \rightarrow -t \quad g(-t) = 1 \]
\[ 1 \]

\[ H(j\omega) = g(\omega) = 1 \]
\[ 1 \]
\[ \omega \]

\[ 2\pi x(\omega) = 2\pi \delta(\omega) \]
\[ 2\pi \]

\[ t \rightarrow \omega; \times 2\pi \]

Difficult to find Fourier transform of \[ x(t) = 1 \] from definition.
Determine the Fourier transform of $x(t) = \cos \omega_0 t$.

1. $\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$
2. $\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$
3. $e^{j\omega_0}$
4. $\frac{j\omega}{\omega_0^2 - \omega^2}$
5. none of the above
Duality of the Fourier transform

We can use duality to show that the Fourier transform of a sinusoidal time function is two delta functions in $\omega$.

Start with the Fourier transform of a shifted delta function in time.

\[ x(t) = \delta(t - \lambda) \quad \leftrightarrow \quad g(\omega) = \int_{-\infty}^{\infty} \delta(t - \lambda) e^{-j\omega t} dt = e^{-j\omega \lambda} \]
**Duality of the Fourier transform**

We can use duality to show that the Fourier transform of a sinusoidal time function is two delta functions in $\omega$.

Start with the Fourier transform of a shifted delta function in time. Then apply duality.

\[
\begin{align*}
x(t) &= \delta(t - \lambda) \\
g(\omega) &= \int_{-\infty}^{\infty} \delta(t - \lambda)e^{-j\omega t} \, dt = e^{-j\omega \lambda}
\end{align*}
\]

Then replace $\lambda$ by $\omega_0$ and use Euler’s equation.

\[
\begin{align*}
x(t) &= \cos \omega_0 t = \frac{1}{2} \left( e^{j\omega_0 t} + e^{-j\omega_0 t} \right) \\
\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)
\end{align*}
\]
Determine the Fourier transform of $x(t) = \cos \omega_0 t$.

1. $\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$
2. $\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$
3. $e^{j\omega_0}$
4. $\frac{j\omega}{\omega_0^2 - \omega^2}$
5. none of the above
Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T} \]

Because the Fourier transform of \( e^{jk\omega_0 t} \) is \( 2\pi \delta(\omega - k\omega_0) \),

\[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0). \]

This expression shows the relation between the Fourier Series and Fourier transform for a periodic signal.
Relation between Fourier Series and Transform

A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} \leftrightarrow \]

\[ \omega \quad 0 \quad \omega_0 \quad 2\pi a_4 \quad 2\pi a_3 \quad 2\pi a_2 \quad 2\pi a_1 \quad 2\pi a_0 \quad 2\pi a_4 \]

\[ a_{-4} a_{-3} a_{-2} a_{-1} a_0 a_1 a_2 a_3 a_4 \]
Check Yourself

What is the Fourier transform of the impulse train

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]
Check Yourself

Impulse train is periodic: find Fourier series representation.

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ a_k = \frac{1}{T} \int T_n \sum_{n=-\infty}^{\infty} \delta(t - nT)e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\omega_0 t} dt = \frac{1}{T} \]

The Fourier transform is then

\[ X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_0) ; \quad \omega_0 = \frac{2\pi}{T} \]

The Fourier transform of an impulse train in time is an impulse train in frequency.
Relations among Fourier Representations

Start with an aperiodic CT signal. Determine its Fourier transform. Convert the signal so that it can be represented by alternate Fourier representations and compare.
Start with the CT Fourier Transform

Determine the Fourier transform of the following signal.

Could calculate Fourier transform from the definition.

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \]

Ugly. Alternatively, could calculate \( x(t) \) by convolution of two square pulses: \( x(t) = (y * y)(t) \).
Check Yourself

The Fourier transform of $y(t)$ is a sinc function (i.e., of the form $\frac{\sin \omega}{\omega}$).

Determine $C$ and $\omega_0$. 

The constant $C$ is the “DC value” (i.e., value at $\omega = 0$) of $y(t)$, which is the area under $y(t)$.

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{j\omega t}dt$$

$$C = Y(j0) = \int_{-\infty}^{\infty} y(t)dt = 1$$
The frequency $\omega_0$ is the lowest frequency for which the Fourier transform is zero.

$$Y(j\omega_0) = 0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j\omega_0 t} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\cos \omega_0 t + j \sin \omega_0 t) dt$$

The lowest frequency for which this integral will be zero occurs when the interval $-\frac{1}{2} < t < \frac{1}{2}$ contains one full period of the sinusoid.

The period must be $T = 1$, so the frequency must be $\frac{2\pi}{1} = 2\pi$. 
Start with the CT Fourier Transform

The Fourier transform of $y(t)$ is a sinc function (i.e., of the form $\frac{\sin \omega}{\omega}$).

If $x(t) = (y \ast y)(t)$ then $X(j\omega) = Y(j\omega) \times Y(j\omega)$. 

\[
\begin{align*}
    y(t) & \quad \leftrightarrow \quad Y(j\omega) \\
    x(t) & \quad \leftrightarrow \quad X(j\omega)
\end{align*}
\]
Determine the Fourier transform of the periodic extension of $x(t)$ to period $T = 4$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$

Could calculate $Z(j\omega)$ for the definition ... ugly.
What is a good way to calculate the Fourier transform of \( z(t) \)?

\[
z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)
\]
Relation between Fourier Transform and Series

We can calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k) = (x * p)(t)$$
We can calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k) = (x * p)(t)$$

What is the impulse train?
We can calculate \( z(t) \) by convolving \( x(t) \) with an impulse train.

\[
z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k) = (x * p)(t)
\]

where

\[
p(t) = \sum_{k=-\infty}^{\infty} \delta(t + 4k)
\]
We can calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k) = (x * p)(t)$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t + 4k)$$

What is the Fourier transform of this impulse train?
Relation between Fourier Transform and Series

We can calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k) = (x \ast p)(t)$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t + 4k)$$

Then

$$Z(j\omega) = X(j\omega) \times P(j\omega)$$

We already know $P(j\omega)$: it’s also an impulse train!
Multiply the Fourier transform of $x(t)$ times the Fourier transform of $p(t)$.

Relation between Fourier Transform and Series

$X(j\omega)$

$P(j\omega)$

$Z(j\omega)$
The Fourier transform of a periodically extended function is a discrete function of frequency $\omega$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$
The weight (area) of each impulse in the Fourier transform of a periodically extended function is $2\pi$ times the corresponding Fourier series coefficient.

$$Z(j\omega)$$

$$\frac{\pi}{2} \quad \frac{\pi}{2}$$

$$a_k$$

$$-1 \quad 1$$
The effect of periodic extension of \( x(t) \) to \( z(t) \) is to sample the frequency representation.

\[ X(j\omega) \]

\[ Z(j\omega) \]

\[ a_k \]

\[ k \]
Relation between Fourier Transform and Series

Periodic extension of a CT signal produces a discrete function of frequency.

Periodic extension

= convolving with impulse train in time
= multiplying by impulse train in frequency
→ sampling in frequency

periodic extension

interpolate sample

periodic extension

interpolate sample

(periodic extension)

\( N \to \infty \)
Sampling a CT signal generates a DT signal.

\[ x[n] = x(nT) \]

Take \( T = \frac{1}{2} \).
Relations between CT and DT transforms

We can generate a signal with the same shape by multiplying $x(t)$ by an impulse train with $T = \frac{1}{2}$.

$$x_p(t) = x(t) \times p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT)$$
Relations between CT and DT transforms

Multiplying $x(t)$ by an impulse train in time is equivalent to convolving $X(j\omega)$ by an impulse train in frequency (then $\div 2\pi$).

$$X(j\omega)$$

$\omega$

$-2\pi$ $2\pi$

$P(j\omega)$

$\omega$

$-4\pi$ $4\pi$

$\cdots$ $\cdots$

$X_p(j\omega)$

$\omega$

$-4\pi$ $4\pi$
Relations between CT and DT transforms

The Fourier transform of the “sampled” signal $x_p(t)$ is periodic in $\omega$ with period $4\pi$.

\[
x_p(t)
\]

\[
X_p(j\omega)
\]
Relations between CT and DT transforms

The Fourier transform of the “sampled” signal $x_p(t)$ has the same shape as the DT Fourier transform of $x[n]$. 

\[ X(e^{j\Omega}) \]
DT Fourier transform

The CT Fourier transform of a “sampled” signal ($x_p(t)$) is equal to the DT Fourier transform of the samples ($x[n]$) where $\Omega = \omega T$, i.e., $X(j\omega) = X(e^{j\Omega})|_{\Omega=\omega T}$.

The DT Fourier transform

$$X(e^{j\Omega}) = \sum_{n=\infty}^{\infty} x[n]e^{-j\Omega n} \quad \text{ (“analysis” equation)}$$

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\Omega})e^{j\Omega n} d\Omega \quad \text{ (“synthesis” equation)}$$

CT Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \text{ (“analysis” equation)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \text{ (“synthesis” equation)}$$
**DT Fourier transform**

The CT Fourier transform of a “sampled” signal \( x_p(t) \) is equal to the DT Fourier transform of the samples \( x[n] \) where \( \Omega = \omega T \), i.e., \( X(j\omega) = X(e^{j\Omega}) \bigg|_{\Omega=\omega T} \).

\[
\Omega = \omega T = \frac{1}{2} \omega
\]
The CT Fourier transform of a “sampled” signal \((x_p(t))\) is equal to the DT Fourier transform of the samples \((x[n])\) where \(\Omega = \omega T\), i.e., \(X(j\omega) = X(e^{j\Omega})\big|_{\Omega=\omega T}\).
Periodic extension of a DT signal is equivalent to convolution of the signal with an impulse train.

\[ x[n] \]

\[ p[n] \]

\[ x_p[n] = (x * p)[n] \]
Relation Between DT Fourier Transform and Series

Convolution by an impulse train in time is equivalent to multiplication by an impulse train in frequency.

\[ X(e^{j\Omega}) \]

\[ P(e^{j\Omega}) \]

\[ X_p(e^{j\Omega}) \]
Relation Between DT Fourier Transform and Series

Periodic extension of a discrete signal \((x[n])\) results in a signal \((x_p[n])\) that is both periodic and discrete. Its transform \((X_p(e^{j\Omega}))\) is also periodic and discrete.

\[
x_p[n] = (x * p)[n]
\]
Relation Between DT Fourier Transform and Series

The weight of each impulse in the Fourier transform of a periodically extended function is $2\pi$ times the corresponding Fourier series coefficient.

$$X_p(e^{j\Omega})$$

$$\cdots$$

$-2\pi$  $-\frac{\pi}{4}$  $\frac{\pi}{4}$  $2\pi$

$$\cdots$$

$$a_k$$

$-8$  $-1$  $1$  $8$

$$\cdots$$

$$\Omega$$

$$\frac{\pi}{2}$$

$$\Omega$$

$$\frac{1}{4}$$
The effect of periodic extension was to sample the frequency representation.

\[ X(e^{j\Omega}) \]

\[ X_p(e^{j\Omega}) \]

\[ a_k \]

\[ \Omega \]

\[ \pi / 2 \]

\[ \frac{1}{4} \]
Relation between Fourier Transforms and Series

Periodic extension of a DT signal produces a discrete function of frequency.

Periodic extension

= convolving with impulse train in time
= multiplying by impulse train in frequency
→ sampling in frequency

\[ N \rightarrow \infty \]

\[ T \rightarrow \infty \]

interpolate sample

periodic DT DTFS
aperiodic DT DTFT

aperiodic CT CTFT

periodic CT CTFS

interpolate sample
Different Fourier representations are related because they apply to signals that are related.

DTFS (discrete-time Fourier series): periodic DT
DTFT (discrete-time Fourier transform): aperiodic DT
CTFS (continuous-time Fourier series): periodic CT
CTFT (continuous-time Fourier transform): aperiodic CT

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**Relations among Fourier Representations**

- DTFS (discrete-time Fourier series): periodic DT
- DTFT (discrete-time Fourier transform): aperiodic DT
- CTFS (continuous-time Fourier series): periodic CT
- CTFT (continuous-time Fourier transform): aperiodic CT

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- **Interactions**:
  - Periodic extension
  - Sampling
  - Interpolation

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- **Diagram**:
  - DTFS → DTFT (N → ∞)
  - CTFS → CTFT (T → ∞)

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- **Key Terms**:
  - Sample
  - Interpolate
  - Periodic Extension