Recitation 22
Sampling and reconstruction

After studying this chapter, you should be able to:
• reconstruct bandlimited signals from their samples, if the samples are frequent enough; and
• explain the factor of 2 in the sampling theorem.

Unless you believe the craziness about string theory, the world is a continuous device. Yet computers are discrete devices. For a computer to represent continuous-time signals usually requires that we sample a signal before handing it to the computer. Sampling turns a continuous-time signal $x(t)$ and into a discrete-time signal $x[n]$ using the recipe

$$x[n] = x(nT),$$

where $T$ is the sampling interval. In general this operation destroys information because it replaces the uncountable infinity of points in $x(t)$ with a countable infinity of points in $x[n]$. A process that destroys information must be irreversible, so it is impossible in general to reconstruct the original signal $x(t)$ from its samples. If $x[n]$ is the unit sample (the impulse), for example, here are several continuous-time signals that pass through those samples:

Requiring that $x(t)$ be continuous eliminates the third candidate but does not remove the ambiguity because the first two candidates are still viable (as are an infinity of others).

Even though continuity of $x(t)$ does not guarantee that sampling preserves information, stronger conditions guarantee that sampling does not destroy information and is therefore reversible. Finding those conditions and recovering a signal from its samples are the themes of this chapter.

Why not forget about sampling and these difficulties and instead agree to process continuous-time signals using only analog hardware like LRC filters? The answer is that it is often faster and cheaper to program a computer than it is to build an analog processor. Therefore, sampling is fundamental to modern engineering.

22.1 When does sampling not destroy information?
Reconstructing $x(t)$ is hopeless if it can wiggle arbitrarily between samples. The first requirement on $x(t)$, therefore, is that it not wiggle too fast. Fast wiggles arise from high frequencies, so $x(t)$ should not contain arbitrarily high frequencies. Therefore, $x(t)$ should be bandlimited: Its Fourier transform should be zero for high-enough frequencies.
The requirement of no arbitrary wiggles is necessary but not sufficient. To find the other necessary conditions, let’s sample a cosine, which is the simplest bandlimited signal. For even more simplicity, we sample \( \cos t \), which is the cosine with the simplest angular frequency (\( \omega = 1 \)).

Here are samples (black dots) from \( \cos t \) using the sampling interval \( T = 0.2 \):

![Samples of \( \cos t \) with \( T = 0.2 \)](image)

If you know that \( \omega = 1 \) is the highest frequency in \( x(t) \), only one reconstruction is possible, namely that \( x(t) = \cos t \).

Using such a tiny sampling interval is overkill for this \( x(t) \). Let’s successively slow down the sampling (increase the interval) until the reconstruction becomes ambiguous. Here is the same cosine sampled using \( T = 0.4 \):

![Samples of \( \cos t \) with \( T = 0.4 \)](image)

And here it is sampled using \( T = 1 \):

![Samples of \( \cos t \) with \( T = 1 \)](image)

In both cases, the samples determine the reconstruction of \( x(t) \), assuming again that the highest frequency is \( \omega = 1 \).

So even \( T = 1 \) is overkill. Let’s therefore increase \( T \) a lot and see whether we can still reconstruct \( x(t) \). Here is \( T = 2\pi \):

![Samples of \( \cos t \) with \( T = 2\pi \)](image)

Oh no: At least two reconstructions are possible. One is \( x(t) = 1 \), which is a zero-frequency cosine, and another is the original signal \( x(t) = \cos t \). The signal \( \cos t \) has another reconstruction – an alias – because its frequency is too high relative to the sampling frequency. Since we overshot the critical sampling interval \( T \), we should reduce \( T \) in order to find the critical \( T \) that avoids aliases.

Here is the figure for a smaller sampling interval \( T = 3\pi/2 \):

![Samples of \( \cos t \) with \( T = 3\pi/2 \)](image)

Alas, it also has two reconstructions:
The problem in the examples with ambiguous reconstruction is that \( x(t) \) wiggles between samples. When the samples are too far apart, \( x(t) \) can wiggle. So the second requirement for reconstruction is frequent sampling. How frequent? The borderline case is \( T = \pi \):

With this sampling interval, \( x(t) \) tries to wiggle (change direction) but the next sample arrives just before \( x(t) \) can do so, thereby constraining the reconstruction to be \( \cos t \).

Exercise 100. Can you reconstruct any signal with frequency \( \omega = 1 \) using the borderline sampling interval \( T = \pi \)?

Sampling at exactly this borderline interval can destroy information. Try sampling the shifted cosine \( \cos(t - \pi/2) \) using \( T = \pi \):

The samples are all zero! To avoid this degenerate case, use \( T < \pi \). Therefore the second requirement for reconstruction is to sample at more than twice the frequency of the cosine. This conclusion works for signals with more than one frequency, becoming: To preserve information, sample at more than twice the highest frequency in the signal.

This sampling theorem is most easily proved in the Fourier representation, which is done in Lecture 22. But the argument using the time representation gives a complementary intuition for why the theorem is true and for why it contains that otherwise mysterious factor of \( 2\pi \).

## 22.2 Procedure for sampling and reconstruction

Sampling converts a continuous-time signal into discrete-time signal, and reconstruction converts a discrete-time signal into a continuous-time signal. Sampling followed by reconstruction therefore turns a continuous-time signal into another continuous-time signal. If sampling destroys information in the starting signal, then the starting and ending signals will not be identical. We will study the effect of sampling by comparing the starting and ending signals. Sampling itself is a simple operation, at least mathematically, so the interesting changes happen in the reconstruction operation.

Reconstruction has two conceptual steps. The first step is to turn the discrete-time sequence \( x[n] \) into a continuous signal. A natural way to do so is to replace each sample with a delta function at \( t = nT \) and area \( x[n] \). The intermediate continuous-time signal is then
\[
\sum_n x[n] \delta(t - nT) = x(t) \times \text{time comb with spacing } T.
\]

The resulting signal has an infinite number of infinite discontinuities, so it contains all frequencies. Worse, their amplitudes do not fall to zero even when \(\omega\) goes to infinity. In other words, this signal is as far from being bandlimited as a signal can be.

The second step in the reconstruction is to smooth this nasty signal to recover an approximation to the original signal. Smoothing means to deemphasize high frequencies. In bandlimited reconstruction, which is based on the sampling theorem, we take the extreme approach: Completely remove the high frequencies. If the sampling interval is \(T\), then – following the procedure in Section 22.1 for reconstructing \(\cos t\) – we discard frequencies outside the range \([-\omega_s/2, \omega_s/2]\), where \(\omega_s = 2\pi/T\) is the (angular) sampling frequency. The resulting low-pass-filtered signal is the reconstructed signal.

Let’s compare this Fourier analysis of sampling of reconstruction to reconstructing \(\cos t\) using the Force as we did in Section 22.1: i.e. by looking at the samples and drawing the cosine that goes through them.

Multiplying \(\cos t\) by the time comb convolves the frequency representation of \(\cos t\) with the frequency representation of the time comb. The frequency representation of the time comb is a frequency comb: a train of delta functions in frequency. The starting signal \(\cos t\) has frequency representation

\[
x(\omega) = \pi\delta(t - 1) + \pi\delta(t + 1),
\]

Convolving \(x(\omega)\) with a frequency comb copies the two delta functions over and over at ever higher frequencies. The important conclusion of all this comb gymnastics is that the sampled continuous-time signal has a periodic frequency representation. This consequence is familiar from Section 21.3 where we compared the discrete-time Fourier transform with the continuous-time Fourier transform of a sampled signal.

In the second step of reconstruction, we low-pass filter this sampled continuous-time signal by discarding all but the first period of its frequency representation. This operation leaves us the original signal \(x(t) = \cos t\), as long as the sampling interval is small enough. How small is small enough? We have an answer from the sampling theorem, but let’s find the same answer using this frequency-representation analysis. To do so, imagine increasing the sampling interval until a problem appears. As the sampling interval \(T\) is increases, the frequency comb’s teeth move closer. Convolving with a frequency comb causes a problem when its tooth spacing is smaller than the width of \(x(\omega)\), where the width means the width of the nonzero region of \(x(\omega)\). When that happens, the copies overlap, which is the frequency representation of aliasing, which destroys information about the original signal.

For \(x(t) = \cos t\), the maximum frequency is \(\omega_{\text{max}} = 1\) so the width is 2 because \(x(t)\) (like any real signal) contains positive and negative frequencies. Since the tooth spacing for the frequency comb is \(2\pi/T\), the condition for accurate reconstruction (equivalently, for no aliasing) is that \(2\pi/T > 2\) or \(T < \pi\), which is the critical sampling interval that we found in Section 22.1 and is illustrated again in this figure.
22.3 Impulse response of reconstruction

A cosine is a simple and therefore useful signal with which to understand sampling and reconstruction. It is the extreme case of a signal with the most peaked Fourier transform. The other extreme is the function with the flattest Fourier transform, which is the most peaked time signal \( x(t) = \delta(t) \). Rather than look directly at sampling and reconstructing \( \delta(t) \), we’ll look at a related but more useful case of reconstructing the most peaked discrete-time signal. We are skipping over the sampling operation and starting with \( x[n] \) because sampling – turning the continuous-time signal \( x(t) \) into \( x[n] \) – is such a trivial operation mathematically; engineering it is very hard, but that problem is another story. The only mathematical choice in sampling is the sampling interval \( T \). The mathematically interesting operation is reconstruction, and many methods are possible. This chapter focuses on bandlimited reconstruction, so let’s investigate that method.

**Pause to try 80.** Is sampling followed by bandlimited reconstruction a linear operation?

Sampling – turning \( x(t) \) into \( x[n] \) – is a linear operation. What about reconstruction? Bandlimited reconstruction means multiplying the frequency representation by a pulse to throw out the high frequency copies. Reconstruction therefore convolves the sampled function with the time representation of a frequency pulse. Convolution is a linear operation, so reconstruction is also a linear operation. Therefore, the combination of sampling followed by bandlimited reconstruction is also a linear operation! The exclamation mark is because the combined operation uses multiplication in time and in frequency, and one of those multiplications could have made the whole operation nonlinear. But each multiplication is by a function independent of the signal, so the operations are still linear.

**Pause to try 81.** Is sampling followed by bandlimited reconstruction a time-invariant operation?

As long as the sampling does not destroy information – meaning that no frequencies alias – then sampling followed by bandlimited reconstruction exactly reconstructs the starting signal, whether or not it is shifted in time. So the combined operation of sampling and reconstruction is time invariant if the signal is properly bandlimited. In general, however, the combined operation is not time invariant: For example, reconstructing the shifted cosine in Section 22.1 using \( T = \pi \) produces \( x(t) = 0 \) because sampling at slightly too low a rate destroyed information.

To probe bandlimited reconstruction, we find how it reconstructs the most peaked discrete-time signal, which is the unit sample \( \delta[n] \) (also known as the discrete-time impulse). We are therefore computing the impulse response of the reconstruction operation. An impulse has a flat Fourier transform. Low-pass filtering it by chopping off all frequencies beyond a cutoff frequency convolves the time representation with a sinc function, where

\[
sinc x \equiv \frac{\sin x}{x}.
\]

To avoid mistakes with \( 2\pi \) and \( T \), we’ll guess the amplitude and \( t \)-axis scaling of the sinc rather than compute it directly. A reasonable requirement on the reconstructed signal \( x_r(t) \) is that it at least pass through the samples \( x(nT) \). So the reconstruction of \( \delta[n] \) should be 1 at \( t = 0 \) and 0 at
\[ t = nT \text{ for } n \neq 0. \] A properly scaled sinc function such as \( \alpha \text{sinc} \beta t \) can meet both requirements. Since \( \text{sinc} 0 = 1 \), the requirement that \( x(0) = 1 \) means \( \alpha = 1 \). The zeros of \( \text{sinc} \beta t \) are the zeros of \( \sin \beta t \) (except at \( t = 0 \)). Those zeros are \( t = n\pi/\beta \) for \( n \neq 0 \). To put the zero crossings of the sinc at \( nT \) requires \( \pi/\beta = T \) or \( \beta = \pi/T \). So the reconstruction of the unit sample is

\[ x_r(t) = \text{sinc} \frac{\pi t}{T}, \]

which looks like

The unit-sample response is useful because it gives a reconstruction procedure starting with a general discrete-time signal.

**Exercise 101.** Here is a continuous-time signal: \( x(t) \) is \( 1 - |t| \) for \( |t| \leq 1 \) and is zero elsewhere. Sample it with \( T = 1 \) and obtain its reconstruction \( x_r(t) \). Sketch \( x(t) \) and \( x_r(t) \) and mark the samples.

**Exercise 102.** Here is another continuous-time signal: \( x(t) \) is \( 1 - t^2 \) for \( |t| \leq 1 \) and is zero elsewhere. Sample it with \( T = 1 \) and obtain its reconstruction \( x_r(t) \). Sketch \( x(t) \) and \( x_r(t) \) and mark the samples.

**Exercise 103.** Compare the two preceding answers and explain the comparison in terms of frequency representations.

**Exercise 104.** Here is a more interesting continuous-time signal: \( x(t) \) is 1 for \( |t| \leq 3/2 \) and is zero elsewhere. Sample it with \( T = 1 \) and obtain its reconstruction \( x_r(t) \). Sketch \( x(t) \) and \( x_r(t) \) and mark the samples.