Lecture 2
Multiple Representations of Discrete-Time Systems
Multiple Representations of Discrete-Time Systems

Discrete-time systems can be represented in a variety of ways.

Verbal description: ‘To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.’

Difference equation:

\[ y[n] = x[n] - x[n-1] \]

Block diagram:

Same input-output behavior, different strengths/weaknesses:

- **word statements** preserve underlying physics.
- **difference equations** are mathematically compact.
- **block diagrams** illustrate signal flow paths.
Step-by-Step solutions

Block diagrams and difference equations are convenient for step-by-step analysis. Let $x[n]$ equal the “unit sample” $\delta[n]$,

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise}. \end{cases}$$

Using the recursion:

$$y[n] = x[n] - x[n - 1]$$

$$y[0] = x[0] - x[-1] = 1 - 0 = 1$$

$$y[1] = x[1] - x[0] = 0 - 1 = -1$$


$$\cdots$$
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\]

\[
\]

\[
\]

\[ \cdots \]

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x[n] = \delta[n]
\]

\[
y[n]
\]
Block diagrams and difference equations are convenient for step-by-step analysis. Let $x[n]$ equal the “unit sample” $\delta[n]$, where

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Step-by-Step solutions

Using the block diagram. Start “at rest.”

\[ x[n] = \delta[n] \]

\[ y[n] \]
Using the block diagram. Start “at rest.”

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Step-by-Step solutions

Using the block diagram. Start “at rest.”

$x[n] = \delta[n]$  

$y[n]$
An alternative **operator** approach focuses on signals rather than samples.

Operator approach: nodes represent whole signals (e.g., $X$ and $Y$) and boxes **operate** on those signals:

- **Delay** = shift whole signal to right 1 time step
- **Add** = sum two signals
- **$-1$**: invert whole signal
Symbols can compactly represent diagrams.

Let $\mathcal{R}$ represent the right-shift operator:

$$Y = \mathcal{R}\{X\} \equiv \mathcal{R}X$$

where $X$ represents the whole input signal ($x[n]$ for all $n$) and $Y$ represents the whole output signal ($y[n]$ for all $n$).

Representing the difference machine

with $\mathcal{R}$ leads to the equivalent representation

$$Y = X - \mathcal{R}X = (1 - \mathcal{R}) X$$
Let $Y = R X$. Which of the following are true:

1. $y[n] = x[n]$ for all $n$
2. $y[n + 1] = x[n]$ for all $n$
3. $y[n] = x[n + 1]$ for all $n$
4. $y[n - 1] = x[n]$ for all $n$
5. none of the above
System operations have simple operator representations.

Cascade systems $\rightarrow$ multiply operator expressions.

Using operator notation:

$$Y_1 = (1 - R) X$$

$$Y_2 = (1 - R) Y_1$$

Substituting for $Y_1$:

$$Y_2 = (1 - R)(1 - R) X$$
Operator Algebra

Operator expressions expand and reduce like polynomials.

Using difference equations:
\[ y_2[n] = y_1[n] - y_1[n-1] \]
\[ = (x[n] - x[n-1]) - (x[n-1] - x[n-2]) \]
\[ = x[n] - 2x[n-1] + x[n-2] \]

Using operator notation:
\[ Y_2 = (1 - R) Y_1 = (1 - R)(1 - R) X \]
\[ = (1 - R)^2 X \]
\[ = (1 - 2R + R^2) X \]
Operator Approach

Applies your existing expertise with polynomials to understand block diagrams, and thereby understand systems.
Operator Algebra

Operator notation facilitates seeing relations among systems. “Equivalent” block diagrams (assuming both initially at rest):

Equivalent operator expression:

\[(1 - R)(1 - R) = 1 - 2R + R^2\]
Operator notation prescribes operations on signals, not samples: e.g., start with $X$, subtract 2 times a right-shifted version of $X$, and add a double-right-shifted version of $X$!

$X:$

$-2RX:$

$+R^2X:$

$y = X - 2RX + R^2X:$
Expressions involving $\mathcal{R}$ obey many familiar laws of algebra, e.g., commutativity.

$$\mathcal{R}(1 - \mathcal{R})X = (1 - \mathcal{R})\mathcal{R}X$$

This is easily proved by the definition of $\mathcal{R}$, and it implies that cascaded systems commute (assuming initial rest).
Operator Algebra

Multiplication distributes over addition.

Equivalent systems

Equivalent operator expression:

\[ R(1 - R) = R - R^2 \]
The associative property similarly holds for operator expressions.

Equivalent systems

Equivalent operator expression:

\[
(1 - R)R (2 - R) = (1 - R) (R(2 - R))
\]
Operator Algebra: Explicit and Implicit Rules

Recipes versus constraints.

\[ Y = (1 - R) X \]

Recipe: output signal equals difference between input signal and right-shifted input signal.

Constraint: find the signal \( Y \) such that the difference between \( Y \) and \( RY \) is \( X \). But how?
Example: Accumulator

Try step-by-step analysis: it always works.

\[ x[n] \rightarrow + \rightarrow y[n] \]

Find \( y[n] \) given \( x[n] = \delta[n] \):

\[
\begin{align*}
    y[0] &= x[0] + y[-1] = 1 + 0 = 1 \\
    y[1] &= x[1] + y[0] = 0 + 1 = 1 \\
    \vdots
\end{align*}
\]

\( x[n] = \delta[n] \)

\( y[n] \)

Persistent response to a transient input!
**Example: Accumulator**

The response of the accumulator system could also be generated by a system with infinitely many paths from input to output, each with one unit of delay more than the previous.

\[ Y = (1 + R + R^2 + R^3 + \cdots) X \]
**Example: Accumulator**

These systems are equivalent in the sense that if each is initially at rest, they will produce identical outputs from the same input.

\[(1 - R) Y_1 = X_1 \iff Y_2 = (1 + R + R^2 + R^3 + \cdots) X_2\]

**Proof:** if \( X_2 = X_1 \) then \( Y_2 = Y_1 \)

\[
Y_2 = (1 + R + R^2 + R^3 + \cdots) X_2 \\
= (1 + R + R^2 + R^3 + \cdots) X_1 \\
= (1 + R + R^2 + R^3 + \cdots) (1 - R) Y_1 \\
= ((1 + R + R^2 + R^3 + \cdots) - (R + R^2 + R^3 + \cdots)) Y_1 \\
= Y_1
\]
Example: Accumulator

The system functional for the accumulator is the reciprocal of a polynomial in $\mathcal{R}$.

\begin{align*}
(1 - \mathcal{R}) Y &= X \\
\text{The product } (1 - \mathcal{R}) \times (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots) \text{ equals 1.}
\end{align*}

Therefore the terms $(1-\mathcal{R})$ and $(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots)$ are reciprocals.

Thus we can write

\[
\frac{Y}{X} = \frac{1}{1 - \mathcal{R}} = 1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \mathcal{R}^4 + \cdots
\]
Example: Accumulator

The reciprocal of $1 - R$ can also be evaluated using synthetic division.

\[
\begin{array}{c|cccc}
 & 1 & +R & +R^2 & +R^3 & +\cdots \\
\hline
1 - R & 1 & -R \\
\hline
 & R & \multicolumn{4}{c}{R - R^2} \\
\hline
 & R^2 & \multicolumn{4}{c}{R^2 - R^3} \\
\hline
 & R^3 & \multicolumn{4}{c}{R^3 - R^4} \\
\hline
 & & & & & \cdots
\end{array}
\]

Therefore

\[
\frac{1}{1 - R} = 1 + R + R^2 + R^3 + R^4 + \cdots
\]
Multiple Representations: Check Yourself

System #1 is represented by the following functional:

\[
\frac{Y}{X} = \frac{1 - \frac{1}{2}R}{1 - R}.
\]

How many of the following systems are equivalent to System #1 (provided they are all initially at rest)?
Multiple Representations of Discrete-Time Systems

Now you know four representations of discrete-time systems.

**Verbal descriptions:** preserve the underlying physics.

“To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences.”

**Difference equations:** mathematically compact.

\[ y[n] = x[n] - x[n - 1] \]

**Block diagrams:** illustrate signal flow paths.

![Block Diagram](image)

**Operator representations:** analyze systems as polynomials.

\[ Y = (1 - R) X \]