Energy and Noise

- Signal-to Noise Ratio (SNR)
- Constellation diagrams and SNR
- Bit error rate versus SNR
Review of Digital Modulation

- Transmitter sends discrete-valued signals over an analog communication channel
- Receiver samples recovered baseband signal
  - Noise and ISI corrupt received signal
- Key techniques
  - Properly design transmit and receive filters for low ISI
  - Sample and slice received signals to detect symbols
Definition of Signal-to-Noise Ratio

- Signal-to-Noise ratio (SNR) indicates impact of noise on system performance (Ratio of Signal power to Noise power)

\[
SNR = \frac{\tilde{P}_{signal}}{\tilde{P}_{noise}}
\]

- We often like to use units of \( \text{dB} \) to express SNR:

\[
SNR \ (\text{dB}) = 10 \log \left( \frac{\tilde{P}_{signal}}{\tilde{P}_{noise}} \right)
\]
SNR Example

- Scaling the gain factor $A$ leads to different SNR values
  - Lower $A$ results in lower SNR
  - Signal quality steadily degrades with lower SNR

SNR Example

SNR = 20.4 dB

SNR = 10.7 dB

SNR = 0.4 dB
Impact of SNR on Receiver Constellation

- SNR influenced by transmitted power, distance between transmitter and receiver, and noise
Impact of Increased Signal on Constellation

- Increase in received signal power leads to increased separation between symbols
  - SNR is improved if noise level unchanged
Experiment to see Statistical Distribution

- Create histograms of sample values from trials of increasing lengths
- Assumption of independence and stationarity implies histogram should converge to a shape known as a probability density function (PDF)
The Probability Density Function PDF

- Define \( x \) as a random variable whose PDF has the same shape as the histogram we just obtained.

- Denote PDF of \( x \) as \( f_X(x) \)
  - Scale \( f_X(x) \) such that its overall area is 1

\[
\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1
\]

This shape is referred to as a Gaussian PDF.
Formalizing Probability

- The *probability* that random variable $x$ takes on a value in the range of $x_1$ to $x_2$ is calculated from the PDF of $x$ as:

$$\text{Prob}(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_X(x) \, dx$$

- Note that probability values are always in the range of 0 to 1
  - Higher probability values imply greater likelihood that the event will occur
Example Probability Calculation

- Verify that overall area is 1:

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = \int_{0}^{2} 0.5 \, dx = 1
\]

- Probability that \( x \) takes on a value between 0.5 and 1.0:

\[
\text{Prob}(0.5 \leq x \leq 1.0) = \int_{0.5}^{1.0} 0.5 \, dx = 0.25
\]
Mean and Variance

- The mean of random variable $x$, $\mu_x$, corresponds to its average value
  - Computed as
  \[
  \mu_x = \int_{-\infty}^{\infty} x f_X(x) \, dx
  \]

- The variance of random variable $x$, $\sigma_x^2$, gives an indication of its variability
  - Computed as
  \[
  \sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) \, dx
  \]

- The standard deviation of a random variable $x$, is denoted $\sigma_x$
Visualizing Mean and Variance from PDF

- Changes in mean shift the center of mass of PDF
- Changes in variance narrow or broaden the PDF
  - Note that area of PDF must always remain equal to one
Example Mean and Variance Calculation

- **Mean:**
  \[\mu_x = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{2} x \frac{1}{2} \, dx = \frac{1}{4} x^2 \bigg|_{0}^{2} = 1\]

- **Variance:**
  \[\sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) \, dx = \int_{0}^{2} (x - 1)^2 \frac{1}{2} \, dx\]
  \[= \frac{1}{6} (x - 1)^3 \bigg|_{0}^{2} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}\]
Quantifying the Impact of Noise

- Minimum separation between symbols: $d_{\text{min}}$
- PDF of noise: zero mean Gaussian PDF
  - Variance of noise sets the spread of the PDF
- Bit errors: occur when noise moves a symbol by a distance more than $d_{\text{min}}/2$
Impact of Reduced SNR (Reduced Signal)

- Lower SNR leads to a reduced value for $d_{\text{min}}$
- Leads to a higher bit error rate
  - Assumes noise variance is unchanged
Impact of Symbol Reduction

- Reducing the number of symbols leads to an increased value for $d_{\text{min}}$
- Leads to a lower bit error rate
  - Assuming SNR remains constant
Can We Estimate Bit Error Rate?

- **Bit Error Rate** depends on:
  - **SNR**
    - Received signal power versus noise variance
  - **Number of constellation points**
    - Sets $d_{\text{min}}$ at a given level of received signal power
Let's Start with a Detailed System View

- Assumptions: No ISI, 4-point constellation
A Closer Examination of Signal and Noise

Communication Channel

Baseband Input

\[ i_N(t) \rightarrow i_N(t) \]

\[ q_N(t) \rightarrow q_N(t) \]

Transmit & Receive Pair

\[ d_{\text{min}} \]

Receiver Output

\[ d_{\text{min}} \]

Sample & Slice

\[ l_{\text{OUT}} \rightarrow \]

\[ Q_{\text{OUT}} \]

Communication Channel for Q Channel

\[ Q_{\text{IN}} \rightarrow \]

\[ d_{\text{min}} \]

\[ Q_{\text{received}} \]

Decision Boundary

\[ d_{\text{min}}/2 \]

\[ Q_{\text{signal}} \]

\[ d_{\text{min}}/2 \]

\[ Q_{\text{noise}} \]

Slicer

\[ 1 \rightarrow \]

\[ 0 \rightarrow \]

\[ Q_{\text{OUT}} \]

PDF of Noise

\[ f_X(x) \]

Variance

\[ \sigma^2 \]

\[ x \]

1

0

-1
The Binary Symmetric Channel Model

- Provides a binary signaling model of channel
**Computation of SNR**

\[
\Rightarrow \quad SNR(dB) = 10 \log \left( \frac{\left(\frac{d_{\text{min}}}{2}\right)^2}{\sigma^2} \right)
\]

**Communication Channel for Q Channel**

- **PDF of Received Q Sample**
  - Transmitted 0: \( f_{Q_0}(y) \)
  - Transmitted 1: \( f_{Q_1}(x) \)
  - Probability of Bit Error \( = P_e \)
  - Decision Boundary: \( \frac{d_{\text{min}}}{2} \)

**Signal Variance**
\[
= \left(\frac{d_{\text{min}}}{2}\right)^2
\]

**Noise Variance**
\[
= \sigma^2
\]
Resulting Bit Error Rate Versus SNR

Note:
- Bit Error Rate = $P_e$
- $SNR \ (dB) = 10 \log \left( \frac{(d_{min}/2)^2}{\sigma^2} \right)$
- Gaussian PDF for noise
Summary

• Constellation diagrams allow intuitive approach of quantifying *uncoded* bit error rate of a channel
  - Function of SNR and number of constellation points

• A digital communication channel can be viewed in terms of a binary signaling model
  - Focuses attention on key issue of bit error rate

• For a given number of bits/symbol the bit error rate decreases as SNR increases.