Intro to Fourier Series

- Function decomposition
- Even and Odd functions
- Fourier Series definition and examples

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Can We Decompose Functions?

- Consider a periodic function such as a square wave

- Can we decompose the above waveform into a weighted sum of basis functions?
- What is a good choice for such basis functions?
- How do we calculate the weights?
Consider Sine Wave Basis Functions

- Suppose we consider sine waves of progressively increasing frequencies as our basis functions.
Issue: Sine Waves Are Limited

- A sine wave corresponds to an *odd* function

\[ \sin(\omega_0 t) \]

- Odd function definition:
  \[ f(t) = -f(-t) \]

- Adding odd functions together can *only* produce an odd function
Consider Cosine Wave Basis Functions

- Even function definition:
  \[ f(t) = f(-t) \]

- Cosine waves are even functions
Combine Cosines and Sines

- If we use both cosine and sine waveforms as basis functions, we can realize both even and odd functions (and any combination)

**Even**

- \( \cos(\omega_0 t) \)
- \( \cos(2\omega_0 t) \)
- \( \cos(3\omega_0 t) \)

**Odd**

- \( \sin(\omega_0 t) \)
- \( \sin(2\omega_0 t) \)
- \( \sin(3\omega_0 t) \)
The Fourier Series

- A periodic waveform, $x(t)$, with period $T$ can be represented as an infinite sum of weighted cosine and sine waveforms

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_0 t) + b_n \sin(nw_0 t)$$

where for $n > 0$:

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(nw_0 t) \, dt, \quad b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(nw_0 t) \, dt$$

and where:

$$w_0 = \frac{2\pi}{T}, \quad a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, dt$$
Sine Wave Example

\[ x(t) = K \sin(w_0 t) \]

\[ \omega_0 = \frac{2\pi}{T} \]

\[ a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} K \sin(w_0 t) \, dt = 0 \quad \text{(DC Average is 0)} \]

\[ a_n = \frac{2}{T} \int_{t_o}^{t_o+T} K \sin(w_0 t) \cos(nw_0 t) \, dt \]

\[ = \frac{K}{T} \int_{t_o}^{t_o+T} \sin((n-1)w_0 t) + \sin((n+1)w_0 t) \, dt = 0 \]

\[ b_n = \frac{2}{T} \int_{t_o}^{t_o+T} K \sin(w_0 t) \sin(nw_0 t) \, dt \]

\[ = \frac{K}{T} \int_{t_o}^{t_o+T} \cos((n-1)w_0 t) + \cos((n+1)w_0 t) \, dt = \begin{cases} K & (n = 1) \\ 0 & (n > 1) \end{cases} \]
Graphical View of Fourier Series (Sine)

\[ x(t) = K \sin(\omega_0 t) \]

- We can plot Fourier coefficients as a function of index or frequency

\[ \omega_0 = \frac{2\pi}{T} \]
Fourier Series of Cosine

\[ x(t) = K \cos(\omega_o t) \]

\[ \omega_o = \frac{2\pi}{T} \]
Fourier Series of Cosine with DC component

\[ x(t) = K \cos(w_0 t) + K \]

Diagram showing the Fourier series coefficients for a cosine function with DC component.
Fourier Series of Phase-Shifted Cosine

\[ x(t) = K \cos(w_0 t - \theta) \]

- Using a well known trigonometric identity:

\[
K \cos(w_0 t - \theta) = K \cos(\theta) \cos(w_0 t) + K \sin(\theta) \sin(w_0 t)
\]
Vector View of Phase-Shifted Cosine

\[ x(t) = K \cos(\omega_0 t - \theta) \]

- Using a well known trigonometric identity:
  \[
  K \cos(\omega_0 t - \theta) = K \cos(\theta) \cos(\omega_0 t) + K \sin(\theta) \sin(\omega_0 t)
  \]

\[ Q = \sin(\omega_0 t) \]

\[ a_n = K \cos \theta \]
\[ b_n = K \sin \theta \]
Square Wave Example

• By inspection:
  - DC average = 0 \implies a_0 = 0
  - x(t) is odd \implies a_n = 0 \quad (n \neq 1)

\[
b_n = \frac{2}{T} \left( \int_{-\frac{T}{2}}^{0} -A \sin(n\omega_o t) dt + \int_{0}^{\frac{T}{2}} A \sin(n\omega_o t) dt \right)
\]

\[
= \frac{4A}{T} \int_{0}^{\frac{T}{2}} \sin(n\omega_o t) dt = \frac{4A}{T} \frac{1}{n\omega_o} (-\cos(n\omega_o T/2) + 1)
\]

\[
= \frac{4A}{T} \frac{T}{n2\pi} (-\cos(n \frac{2\pi}{T} T/2) + 1) = \frac{2A}{n\pi} (-\cos(n\pi) + 1)
\]

\[
\Rightarrow b_n = \frac{4A}{n\pi} \quad \text{for n odd}, \quad b_n = 0 \quad \text{for n even}
\]
Summary

• Fourier Series decomposes periodic waveforms into an infinite sum of weighted cosine and sine functions
  - We can look at waveforms either in 'time' or 'frequency'
  - Useful tool: even and odd functions

• Some issues we will deal with next time
  - Fourier Series definition covered today is not very compact
    • We will look at a simpler formulation based on complex exponentials
  - Fourier Series only deals with periodic waveforms
    • We will introduce the Fourier Transform to deal with non-periodic waveforms

• Check out the following Java applet demo:
  - Available at: http://www.falstad.com/fourier/