1. (a) We can determine this by inspection.

\[ x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{j\omega_0 t} \]

\[ = 2 - \frac{1}{2} e^{j\omega_0 t} - \frac{1}{2} e^{-j\omega_0 t} + \frac{1}{2} e^{j2\omega_0 t} - \frac{1}{2} e^{-j2\omega_0 t} + 2 e^{j4\omega_0 t} + 2 e^{-j4\omega_0 t} \]

\[ \Rightarrow \hat{X}_n = \begin{cases} 
2 & \text{if } n = 0 \\
-\frac{1}{2} & \text{if } n = \pm 1 \\
-\frac{i}{2} & \text{if } n = \mp 2 \\
2 & \text{if } n = \pm 4 \\
0 & \text{otherwise}
\end{cases} \]

A plot of this would look like:

![Figure 1: Plot of \( \hat{X}_n \)](image)

(b) This is simply a Fourier series calculation.

\[ \hat{X}_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j\omega_0 t} dt \]

\[ = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j\frac{2\pi n t}{T}} dt \]

\[ = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi n t} dt \]

\[ = \int_{-\frac{T}{2}}^{\frac{T}{2}} (2t + 1)e^{-j2\pi n t} dt + \int_{0}^{\frac{T}{2}} (-2t + 1)e^{-j2\pi n t} dt \]

\[ = \left[ (2t + 1) \frac{1}{-2\pi n} e^{-j2\pi n t}|_{0}^{1/2} - \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2}{-j2\pi n} e^{-j2\pi n t} dt \right] \]

\[ + \left[ (-2t + 1) \frac{1}{j2\pi n} e^{-j2\pi n t}|_{1/2}^{0} - \int_{0}^{\frac{T}{2}} \frac{2}{j2\pi n} e^{-j2\pi n t} dt \right] \]

\[ = \frac{1}{-j2\pi n} - \frac{1}{-j2\pi n} + \frac{1}{j2\pi n} - \frac{1}{j2\pi n} e^{-j2\pi n} = \frac{2}{j2\pi n} (e^{j\pi n} - 1) \]
Note that $\hat{X}_0$ is indeterminate in this form. However, we can note
\[
\hat{X}_0 = \int_{-\pi}^{\pi} x(t) dt = \frac{1}{2}.
\]
\[\Rightarrow \hat{X}_n = \begin{cases} 
\frac{1}{2} & \text{if } n = 0 \\
\frac{-2}{(\pi n)^2} & \text{if } n = \text{odd} \\
0 & \text{otherwise}
\end{cases}
\]

(c) We can see that this is just the sum of two pulses centered around 0. Both pulses will have height $\frac{1}{2}$. One will be in the range $(-\frac{1}{4}, \frac{1}{4})$ and the other will be in the range $(-\frac{1}{2}, \frac{1}{2})$. We can note from slide 15 of Lecture 3 that a pulse will have a frequency response of the form
\[
X(f) = \frac{A \sin(2\pi f T_p)}{\pi f}
\]

i. Note that since the two pulses sum in the time domain, they will also sum in the frequency domain.
\[\Rightarrow X(f) = X_1(f) + X_2(f) = \frac{\sin(\pi f/2)}{2\pi f} + \frac{\sin(\pi f)}{2\pi f}
\]

ii. (Optional) $X(f) = A(f) + B(f)$.
\[\Rightarrow A(f) = \frac{\sin(\pi f/2) + \sin(\pi f)}{2\pi f}
\]
\[B(f) = 0
\]

Now we can simply compute $|X(f)| = \sqrt{A(f)^2 + B(f)^2}$ and $\Phi(f) = \tan^{-1}(B(f)/A(f))$.
\[|X(f)| = \left| \frac{\sin(\pi f/2) + \sin(\pi f)}{2\pi f} \right|
\]
\[\Phi(f) = \begin{ cases} 
0 & \text{if } A(f) \geq 0 \\
\pi & \text{if } A(f) < 0
\end{cases}
\]

iii. (Optional) Using Matlab, I found the magnitude of $X(f)$ to be:

![Figure 2: Plot of $|X(f)|$](image)
2. (a) i. We should choose $s_2(t) = 5000\text{Hz}$ and $f_c > 500\text{Hz}$. The plots for $M(f)$, $A(f)$ and $D(f)$ are given in Figure 3.

![Figure 3: Frequency domain view of system](image)

ii. We need to add the same system as Question 3b in HW#2. In this case, we would modulate $y(t)$ by 500Hz and then lowpass at 500Hz, as shown in the figure in the original question. The plots for $M(f)$, $A(f)$ and $D(f)$ are given in Figure 4.

![Figure 4: Frequency domain view of system](image)

(b) The four basic types of filters are shown in Figure 5. We can never achieve such steep cutoffs because we would need an filter with an order of infinity. For example, in the lowpass filter, we know that the difference equation will resemble a sinc (c.f. slide 17 of Lecture 5), which extends for $(-\infty, \infty)$. In practical systems, we discard the terms that are close to zero. This produces a good approximation to the ideal filter, but with some ripples.

![Figure 5: Frequency domain view of system](image)
(c)  i. We know from HW#1 that a pulse in the time domain produces a sinc in the frequency domain. The reverse is also true. In general for a pulse $H_c(f)$ with cutoff $f_c$:

$$h_c(t) = \int_{-\infty}^{\infty} H_c(f)e^{-j2\pi ft}df$$

$$= \int_{-f_c}^{f_c} Ae^{-j2\pi ft}df$$

$$= \frac{A}{j2\pi t} \left[ e^{-j2\pi ft} - e^{j2\pi ft} \right]$$

$$= \frac{A}{j2\pi t} (-2j \sin 2\pi f_c t)$$

Referring to the figure, we can see $H_1(t)$ has $f_c = \frac{1}{4}$ and $H_2(t)$ has $f_c = 1$

$$\Rightarrow h_1(t) = \frac{\sin \pi t/2}{\pi t}$$

$$h_2(t) = \frac{\sin 2\pi t}{\pi t}$$

ii. Note that convolution in the frequency domain is multiplication in the time domain. Thus,

$$h(t) = h_1(t) + h_2(t)$$

$$= \frac{\sin(\pi t/2)}{(\pi t)^2} \sin(2\pi t)$$

iii. $H(f)$ is a more practical filter compared to the ideal $H_2(f)$ because it approaches 0 faster as $t$ gets large. Thus, the terms we ignore when creating a FIR filter is not as significant and the frequency response will feature fewer ripples.

3. (a) Receiver noise and ISI are not directly related to each other. ISI is the result of adding a transmit lowpass filter at the transmitter while receiver noise is the the perturbation of the signal due to the channel. However, an increase of either will directly affect the bit error rate at the receiver. An indirect relationship exists in that the transmitter LP filter is chosen to reduce bandwidth and hence receiver noise, but at the same time increases ISI.

The constellation diagram will have more spread out points in the case of higher ISI or receiver noise (meaning higher bit error rate). The eye diagram will be more closed with higher bit error rate but because of different reasons for ISI and receiver noise. For higher ISI, the transitions between states become slower, meaning smooth curves that close the eye (c.f. slide 18 of Lecture 8). For receiver noise, the transitions between states will become jagged due to the noise, which makes the eye smaller also (c.f. slide 5 of Lecture 9).

(b) If the input is $(-3, 3)$, then

$$y(t) = (-3)2 \cos(2\pi 10^8 t) + (3)2 \sin(2\pi 10^8 t)$$

$$= -3e^{j2\pi 10^8 t} - 3e^{-j2\pi 10^8 t} + \frac{3}{j}e^{j2\pi 10^8 t} - \frac{3}{j}e^{-j2\pi 10^8 t}$$

$$= (-3 - 3j)e^{2\pi 10^8 t} + (-3 + 3j)e^{j2\pi 10^8 t}$$

$$= -\sqrt{18}e^{j\tan^{-1}(1)}e^{j2\pi 10^8 t} - \sqrt{18}e^{j\tan^{-1}(-1)}e^{j2\pi 10^8 t}$$

$$= -6\sqrt{2} \cos(2\pi 10^8 t + \tan^{-1}(1))$$

$$= 6\sqrt{2} \cos(2\pi 10^8 t + \tan^{-1}(1) + \pi)$$
\[ A = 6\sqrt{2} \]
\[ f = 10^8 \]
\[ \phi = \tan^{-1}(1) + \pi \]

4. (a) \( \text{Var}(f_1(x)) > \text{Var}(f_2(x)) > \text{Var}(f_3(x)) \)

The variance can be thought about how much of the “mass” is away from the mean. Clearly, \( f_1(x) \) deviates the most from the mean, followed by \( f_2(x) \).

(b) i. Assume we have a slicing level \( x \) such that \( 3 \leq x \leq 4 \), as shown in Figure 6.

![Figure 6: Frequency domain view of system](image)

We want to choose a slicing level to minimize bit error rate, which can be found as:

\[
\text{bit error rate} = f_3^x \left( \frac{1}{4} \right) dx' + f_x^4 \left( -\frac{x'}{10} + \frac{1}{4} \right) dx'
\]
\[
= \frac{x'}{4} - \frac{x^2}{32} + \frac{x'}{4} \left| x \right| \text{ for } x = 3
\]
\[
= \frac{x}{4} - \frac{3}{4} + \left( \frac{1}{2} + \frac{x^2}{32} - \frac{x}{4} \right)
\]
\[
= \frac{x^2}{32} - \frac{1}{4}
\]

By looking at this function, we can immediately see that this would have a minimum at \( x = 3 \).

\Rightarrow \text{slicing threshold should be set at } 3V

ii. Note that SNR = \( 10 \log \frac{\sigma_S}{\sigma_N} \).

\[
\sigma_S = \sum_s (s - \mu_s)^2 p(S = s)
\]
\[
= \left( \frac{1}{2} \right) \left( 0 - \frac{d_{\text{min}}}{2} \right)^2
\]
\[
= \left( \frac{d_{\text{min}}}{2} \right)^2
\]
\[
\mu_N = \int_{x=-\infty}^{\infty} x f(x) dx
\]
\[
= \int_{x=-\infty}^{x=0} \frac{x}{4} dx + \int_{x=0}^{x=4} \left( -\frac{x^2}{16} + \frac{x}{4} \right) dx
\]
\[
= \frac{1}{6}
\]
\[
\sigma_N = \int_{x=-\infty}^{\infty} (x - \mu_N)^2 f(x) dx
\]
\[
= \int_{x=-\infty}^{x=0} \frac{x^2}{4} dx + \int_{x=0}^{x=4} \left( -\frac{x^2}{16} + \frac{x}{4} \right)^2 dx
\]
\[
= \frac{71}{32}
\]
\[
\Rightarrow \text{SNR} = 10 \log \frac{\sigma_S}{\sigma_N}
\]
\[
= 10 \log \frac{5/2}{71/32}
\]
\[
= 0.518 \text{dB}
\]

iii. From above, bit error rate = \( \frac{x^2}{32} - \frac{1}{4} \) with \( x = 3 \).

\Rightarrow \text{bit error rate} = \frac{1}{32}.