Announcements

- Homework 5 due Mon 11/26
**Shortest Paths**

- how to get from $A$ to $B$ in a graph.
- given digraph $G = (V, E)$ with edge weights described by function $w: E \rightarrow \mathbb{R}$

- define path $p = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$ \{$(v_i, v_{i+1}) \in E \ \forall i$\}
  has weight $w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$

**Example**

- define shortest path from $u$ to $v$
  is path from $u$ to $v$ of min weight $\delta(u, v)$
Well-definedness

(when don't they exist?)

if $G$ has a negative weight cycle

then some shortest paths $u \to v$ do not exist

$\Rightarrow \delta(u,v) = -\infty$

Optimal Substructure

A subpath of a shortest path is a shortest path.

Proof "Cut & paste"

Consider shortest $u \to v$ path

$\Rightarrow \delta(u,v) = -\infty$
Triangle Inequality

For all $u, v, x \in V$: $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$

Proof:

\[
\begin{array}{c}
\text{\textcolor{blue}{\scalebox{1.5}{$\delta(u, v)$}}}
\end{array}
\]

$u \leadsto x \leadsto v$ is a $u \leadsto v$ path

Single source shortest path problem

from given source vertex $s$
find shortest-path weight $\delta(s, v)$ for all nodes $v$
First Attempt:

Assume \( w(u,v) \geq 0 \) for all nodes \( u,v \)

\[ \Rightarrow \] shortest paths exist (if any path exists)

and \( \delta(u,v) \geq 0 \) for all nodes \( u,v \).

Idea: Greedy

0. Maintain set of nodes \( S \) whose shortest-path distance from \( s \) are known

2. At each step, add to \( S \) the node \( v \in V-S \) whose distance estimate is minimum

3. Update distance estimates of nodes adjacent to \( v \)
Dijkstra's Algorithm \((G, w, s)\):

1. \(d[s] \leftarrow 0\)
2. \(d(v) = \text{estimated } \delta(s, v)\) \(\text{Initialize}\)
3. for each \(v \in V - \{s\}\)
   - \(d[v] \leftarrow \infty\)
4. \(S \leftarrow \emptyset\)
5. \(Q \leftarrow V\) \(\leftarrow \text{priority queue keyed on } d[v]\)

while \(Q \neq \emptyset\)

- \(u \leftarrow \text{Extract Min}(Q)\) \(\leftarrow \text{add } u \text{ to } S\)
- \(S \leftarrow S \cup \{u\}\)
- for each \(v \in \text{Adj}[u]\) \(\leftarrow \text{all neighbors of } u\)
  - if \(d[v] > d[u] + w(u, v)\)
    - \(d[v] \leftarrow d[u] + w(u, v)\)
    - \(\Pi[v] \leftarrow u\) \(\text{implicit decrease-key}\)

Example:

\(G (s=A)\)

\(Q: A B C D E\)

\[\begin{array}{cccccc}
0 & \infty & \infty & \infty & \infty \\
10 & 3 & \infty & \infty & \infty \\
11 & 7 & 11 & 11 & \infty \\
\end{array}\]

Shortest-path tree = union of shortest paths from \(s\)

= \(\Sigma \) edge \((v, \Pi[v])\) \(\forall v \in V\)
Correctness

The invariant \( d(v) = \delta(s,v) \) for all nodes \( v \) holds after initialization and any sequence of relaxation steps. (not just Dijkstra)

Proof

Initially, \( d(s) = 0 \), \( d(v) = \infty \) for \( v \neq s \)
\( \delta(s,s) = 0 \), \( \delta(s,v) \leq \infty \)

Suppose (for contradiction) that invariant violated.

Consider first violation \( d(v) \neq \delta(s,v) \)
caused by relaxing \( d(v) \leftarrow d(u) + w(u,v) \)
so \( d(u) + w(u,v) < \delta(s,v) \) must be true at this point too.

but \( \delta(s,v) \leq \delta(s,u) + \delta(u,v) \)
\( \leq \delta(s,u) + w(u,v) \)
\( \leq d(u) + w(u,v) \)

Intuition set \( d \) only if valid path found

\( \Delta \neq \) shortest path <= specific path
\( v \) is 1st violation, so \( u \) can't violate
Correctness Lemma

Suppose \( s \rightarrow \ldots \rightarrow u \rightarrow v \) is a shortest path from \( s \) to \( v \).

If \( d(u) = \delta(s,u) \) and we relax edge \((u,v)\),

then \( d(v) = \delta(s,v) \) after relaxation.

Proof

\[
\delta(s,v) = w(s \rightarrow \ldots \rightarrow u) + w(u,v) \\
= \delta(s,u) + w(u,v)
\]

(by optimal substructure)

Correctness \( I \Rightarrow d[v] \geq \delta(s,v) \)

If \( d(v) = \delta(s,v) \) before relaxation, done.

If \( d(v) > \delta(s,v) \) then

- Relaxation condition satisfied:

\[
\delta(v) > \delta(s,v) = \delta(s,u) + w(u,v) \\
= d(u) + w(u,v) \\
\text{(assumption)}
\]

So set \( d(v) \leftarrow d(u) + w(u,v) \)

\[
= \delta(s,u) + w(u,v) \\
= \delta(s,v)
\]

\( \square \)
**Correctness II**

When Dijkstra terminates,
\[ d(v) = \delta(s, v) \text{ for all } v \in V \]

**Proof:**

\[ d(v) \text{ doesn't change once } v \text{ added to } S. \]

Therefore, suffices to show \( d(v) = \delta(s, v) \)
when \( v \) added to \( S \). (happens eventually!)

Suppose (for contradiction) that \( u \) is first vertex about to be added to \( S \) for which
\[ d(u) \neq \delta(s, u) \]

\[ \Rightarrow d(u) > \delta(s, u) \quad \text{(by Correctness I, also note } u \notin S \text{ since } d(s) = 0, \delta(s, s) = 0) \]

Let \( p \) be shortest path from \( S \) to \( u \) \( \Rightarrow \) \( w(p) = \delta(s, u) \)

Consider first place that \( p \) exits \( S \),
Say via edge \( (x, y) \)

i.e., \( y = \) first vertex along \( p \) in \( V-S \)
(\( \text{which exists because } u \notin S \) yet)

\[ + x = \text{predecessor of } y \text{ along } p \]
(\( \text{which exists because } s \in S \))

\[ u \text{ is first violation } \Rightarrow d(x) = \delta(s, x) \]
when \( x \) added to \( S \), relaxed \( (x, y) \)

**Lemma** \( \Rightarrow d(y) = \delta(s, y) \)
\[ \leq \delta(s, u) \]
\[ < d(u) \quad \text{Subpath, nonnegative weights} \]

also says \( u 
eq y \) \( \Rightarrow \text{contradicts greedy choice of } u \)
Analysis

Initialization

\[
\text{While } Q \neq \emptyset \\
\text{do } u \leftarrow \text{Extract-Min}(Q) \\
S \leftarrow S \cup \{u\} \\
\text{for each } v \in \text{Adj}[u] \\
\text{do if } d(v) > d(w) + w(u, v) \\
\text{then decrease key for } d(v)
\]

\[|V| \text{ times}
\]

\[\leq \text{out-degree}(u) = |E| \text{ decrease keys}
\]

\[\text{time} = \Theta(|V|) \cdot T_{\text{Extract Min}} + \Theta(|E|) \cdot T_{\text{Decrease key}} \quad \text{(same as Prim's)}
\]

| Q \hspace{2cm} T_{\text{Extract Min}} \hspace{2cm} T_{\text{Decrease key}} | \text{Total} |
|---|---|---|---|
| array (unsorted) | \(O(|V|)\) | \(O(1)\) | \(O(|V|^2)\) |
| binary heap | \(O(|\log V|)\) | \(O(|\log V|)\) | \(O(|E|\)v) |