Abstract

Adversarial Routing and Game Theoretic Model

Question: How (as system designers) should we model users’ adversarial behavior?

1 Routing Model

Consider the following Routing Model
- Represent the network as a graph, \( G = (V, E) \) (where we depart from previous network models and assume that the edge set \( E \) is directed)
- Let the capacity of link \((i, j)\) be denoted \( C_{ij} \) (\( \forall (i, j) \in E \) \( C_{ij} > 0 \))
- Denote the rate at which node \( i \) wishes to send data to node \( k \) as \( r_i(k) \)
- Denote a delay function \( D_{ij} : [0, C_{ij}] \to \mathbb{R}^+ \). We make the following assumptions about \( D_{ij} \)
  - \( D_{ij} \) is strictly convex
  - \( D_{ij}(0) > 0, \) \( D_{ij}(0) \geq 0 \)
  - \( \lim_{x \to C_{ij}} D_{ij}(x) = \infty \)

This routing model naturally leads to the following optimization problem

\[
\begin{align*}
\min_{f_{ij} \in \mathbb{R}^+} & \quad \sum_{(i, j) \in E} D_{ij}(f_{ij})f_{ij} \\
\text{s.t.} & \quad x_{ij} \geq 0 \\
& \quad \sum_{j : (i, j) \in E} x_{ij}(k) = r_i(k) + \sum_{l : (l, j) \in E} x_{li}(k) : \forall i \in V, \forall k \in V
\end{align*}
\]  

(1)

Facts

(1) Problem (1) is a convex minimization problem (by the separability of the objective, the convexity of \( xD(x) \), and the linearity of the constraints)

(1a) The optimization therefore has a unique optimum (which can be arrived at using a standard distributed solution mechanism such as the subgradient method)

Ideally, the system designer would like to assign to each node flow rates \( f^*_{ij} \) that solve Problem (1). However, if node \( i \) can alter flow \( f^*_{ij} \) to \( f'_{ij} \) s.t. \( f'_{ij} \) is still feasible and results in smaller cost to node \( i \) (i.e. \( \sum_{j : (i, j) \in E} D_{ij}(f'_{ij})f'_{ij} < \sum_{j : (i, j) \in E} D_{ij}(f^*_{ij})f^*_{ij} \)), then a "rational" node (i.e. one seeking to minimize its own cost) will instead choose to implement \( f'_{ij} \).

2 An Individually Rational Routing Model

We want to understand the effect of greedy behavior of network agents on the global performance. Specifically, we will try to establish that greedy behavior has boundedly suboptimal performance (globally speaking).

Consider the following model, based on individual rationality of the sources:
- For all \( i \) there exist a set of paths \( P_i \)
- For simplicity, assume each node has only one destination \( k \)
Let \( f_p \) be the fraction of times path \( p \) is taken by \( i \to k \) \((p \in \mathcal{P}_i)\).

The basic strategy of each node \( i \) is: if \( f_p > 0 \) and \( D_p < D_p' \), then shift some non-trivial amount of weight from \( f_p \) to \( f_p' \) (where \( D_p \) represents the delay along path \( p \)).

Before we proceed, note that we can rewrite (1) in terms of paths rather than links. Let \( \mathcal{P} = \bigcup_i \mathcal{P}_i \), then Problem (1) is equivalent to

\[
\min_f \sum_{e \in E} D_e(f_e) f_e \\
s.t. \quad f_e = \sum_{p \in P} D_e(f_e) f_e \\
\quad f_p \geq 0 \\
\quad \sum_{p \in \mathcal{P}_i} f_p = r_i
\]

### 2.1 Nash Equilibrium

The Nash Equilibrium is a solution principle in strategic games involving two or more players. Under the Nash Equilibrium no player has incentive to unilaterally deviate from his strategy. Mathematically, let \( D_p(f) = \sum_{e \in p} D_e(f_e) \) where \( f_e = \sum_{p \in P} f_p \). A set of flows \( f = (f_p)_{p \in P} \) on network \( G \) with demand \( r \) is in Nash Equilibrium iff

\[
D_{p_1}(f) \leq D_{p_2}(f) \quad \forall i, \forall p_1, p_2 \in \mathcal{P}_i, \forall \delta \in [0, f_{p_1}]
\]

where

\[
\tilde{f}_p = \begin{cases} f_p - \delta & p = p_1 \\ f_p + \delta & p = p_2 \\ f_p & \text{o.w.} \end{cases}
\]

### 2.2 Wardrop’s Principle

A feasible flow \( f \) for network \( G \) with demand \( r \) with delay function \( D \) is in Nash Equilibrium iff

\[
D_{p_1}(f) \leq D_{p_2}(f) \quad \forall i : p_1, p_2 \in \mathcal{P}_i, f_{p_1} > 0
\]

Stated in words, Wardrop’s principle says that an equivalent characterization to an individual’s route selection strategy being in Nash equilibrium is that for any individual, the delay experienced by any route taken with positive probability is no greater than the delay experienced by any alternate route.

Now, we can state and prove the primary assertion of this section

**Assertion 1** A flow satisfying Wardrop’s Principle will solve the global optimization problem

\[
\min_f \sum_{e \in E} h_e(f_e) \\
s.t. \quad f_e = \sum_{p \in P} D_e(f_e) f_e \\
\quad f_p \geq 0 \\
\quad \sum_{p \in \mathcal{P}_i} f_p = r_i
\]

where \( h_e(f) \triangleq \int_0^f D_e(x)dx \)

(Notice, if \( D_e(x) \) is constant over the domain \( 0 \leq x \leq f_e \) for all \( e \), then Problem (3) is exactly equal to Problem (2)).

To prove the assertion, first note that Problem (3) is a convex minimization (by the positivity of \( D_e(x) \)) and therefore has a unique optimal solution, and that the optimal solution occurs at the point where any direction is an ascent direction.

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Let the solution to (3) be \( \hat{f} \). Because (3) is a convex minimization, there is no \((i,j)\) with path \(p_1, p_2\) and \(0 < \epsilon \leq f_{p_1}\) with \(f_{p_1} > 0\) s.t. \(\sum_e h_e(\hat{f}_e) < \sum_e h_e(f_e)\) where

\[
\hat{f}_p = \begin{cases} 
\hat{f}_p - \delta & p = p_1 \\
\hat{f}_p + \delta & p = p_2 \\
\hat{f}_p & \text{o.w.}
\end{cases} \quad \delta \in (0,\epsilon)
\]

Now, since there is no descent direction, for all \( \hat{f} \)

\[
0 \geq \sum_e h_e(\hat{f}_e) - \sum_e h_e(\hat{f}_e) = \sum_{e \in p_1} h_e(\hat{f}_e) - h_e(\hat{f}_e - \delta) + \sum_{e \in p_2} h_e(\hat{f}_e) - h_e(\hat{f}_e + \delta)
\]

Rearranging and dividing both sides by \( \delta \) gives

\[
\sum_{e \in p_1} h_e(\hat{f}_e) - \sum_{e \in p_2} h_e(\hat{f}_e) \leq \frac{-h_e(\hat{f}_e) + h_e(\hat{f}_e + \delta)}{\delta}
\]

Then taking the limit as \( \delta \to 0 \) gives

\[
\sum_{e \in p_1} h_e(\hat{f}_e) \leq \sum_{e \in p_2} h_e(\hat{f}_e) \iff \sum_{e \in p_1} D_e(\hat{f}_e) \leq \sum_{e \in p_2} D_e(\hat{f}_e) \iff D_{p_1}(\hat{f}) \leq D_{p_2}(\hat{f})
\]

Therefore, using the equivalence between Wardrop’s Principle and the Nash Equilibrium, we have shown the equivalence of the following things:

- \( \hat{f} \) solves (3)
- \( \hat{f} \) is s.t. there d.n.e. \((i,j)\) with path \(p_1, p_2\) and \(0 < \epsilon \leq f_{p_1}\) with \(f_{p_1} > 0\) s.t. \(\sum_e h_e(\hat{f}_e) < \sum_e h_e(f_e)\)
- \( \hat{f} \) obeys Wardrop’s Principle
- \( \hat{f} \) is a Nash Equilibrium

Thus the Assertion is proven.

3 Price of Anarchy

The formulation of individually rational routing begs the question of whether the Nash Equilibrium solution achieves any degree of system optimality. More specifically, can the amount of system-wide utility lost due to individual rationality be bounded. Pursuant that goal, define \( \rho(G, r, D) \triangleq C(f^*) - C(f^{NE}) \).

By the optimality of \( f^* \) for Problem (2), it is immediately evident that \( \rho(G, r, D) \geq 1 \).

**Theorem 1** If \( \exists \alpha \geq 1 \) s.t. \( xD(x) \leq \alpha \int_0^x D(y)dy \). Then

\[
\rho(G, r, D) \leq \alpha
\]

Or, equivalently, \( C(f^{NE}) \leq \alpha C(f^*) \)

Proof:

\[
C(f^{NE}) = \sum_e D_e(f_e^{NE})f_e^{NE} \leq \sum_e \alpha \left( \int_0^{f_e^{NE}} D_e(y)dy \right) = \sum_e h_e(f_e^{NE})
\]

where the inequality is due to the theorem’s assumption. Now, since \( f_e^{NE} \) solves (3) and \( f_e^* \) is feasible for (3) we have

\[
\sum_e h_e(f_e^{NE}) \leq \sum_e h_e(f_e^*) = \alpha \sum_e \int_0^{f_e^*} D_e(y)dy \leq \alpha \sum_e D_e(f_e^*) \int_0^{f_e^*} dy
\]

where the final inequality is due to the fact that \( D(x) \) is monotonically increasing for \( x \geq 0 \). Finally

\[
\alpha \sum_e D_e(f_e^*) \int_0^{f_e^*} dy = \alpha \sum_e D_e(f_e^*)f_e^* = C(f^*)
\]

Combining Equations (4), (5), (6) immediately gives us the result.
4 Summary

We began by formulating a system optimization problem for flow through the network. We then modified the formulation to account for the individual rationality of routers. Next, the Nash Equilibrium and, its equivalent in this context, Wardrop’s Principle were introduced as a solution method for individually rational interactions. It was shown that a Nash Equilibrium solution was equivalent to the system optimality under a modified cost function (Problem (3) vs. Problem (2)). Finally, a theorem was proved relating to bounding the system suboptimality of the individually rational solution method.

There are several outstanding questions regarding individually rational routing. Among them are

1. Is rationality a desirable property? Is it ”right?”
2. Why focus on equilibrium? What is the effect of transients?
3. Is price of anarchy the best measure of system suboptimality? Is the alternative formulation
   \[ \exists \beta > 1 \text{ s.t. } C(f^{NE}; G, r, D) \leq C(f^{*}; G, \beta r, D) \] (essentially a quantification of production loss)
   a better description of the price of anarchy?