The desirable scheduling is max. weight scheduling:

\[ \sigma^*(n) \in \arg\max_{\sigma \in S} \sum_{i=1}^{n} W_i(n) \]

where \( W_i(n) \) is a desirable function of queue size such as \( W_i(n) = Q_i^n(n) \), \( \alpha > 0 \).

The dynamic programming is a general method but can be very expensive computationally and not implementable.

The approximate dynamic programming method based on "tree structure", that is Belief Propagation is excellent method:

when problem structure is simple, i.e.

problem can be solved using natural Linear Prog. relaxation. Such as bipartite graph matching for switch or bipartite graph independent set for wireless network.

Implementation-wise it is terrific due to iterative nature, distributed and simple logical operations.
In general, Belief Propagation can be very poor approximation. Therefore, it is unlikely to serve as efficient implementable method in general.

The “message-passing” paradigm, belief propagation, as a special case of it, is an excellent method for implementable algorithm design. Or more generally, an algorithmic architecture for network problems.

Next, we will present randomized algorithm based on message passing paradigm. This algorithm will be throughput optimal for any ‘switched network’ instance. We will use an example of independent set or wireless network model along with switch model (matching).
Randomization, philosophy: base decision on a random subset of elements rather than the whole space.

Switch:

Choose $d$ matching at random, each time.
Use the heaviest of these $d$ matching to schedule each time.

Theorem. The algorithm is not throughput optimal for $d < x n$ for an $n$-port switch, for $x > 0$.

Proof. Consider an edge $(i,j)$. Under selection of $d$ matchings, the probability that it is not chosen is $(1 - \frac{1}{n})^d$.

For $d < x n$, this is $\geq (1 - \frac{1}{n})^{xn} \approx \exp(-xn)$.

Thus, rate allocated to $(i,j)$ is at most $1 - \exp(-x) < 1$.

Thus, for $\lambda_{ij} = 1 - \frac{1}{2} \exp(-x)$ the algorithm is unstable.
In summary, naive randomized algorithm is not throughput optimal even for $d \gg n$ samples. And, need something ‘clever’.

Algorithm (with memory).

1. Let $\sigma(2)$ be schedule at time $2$.

2. At time $t+1$:
   a. Choose a schedule at random from $S$, call it $R(2n)$.
   b. $\sigma(2n) = \text{argmax} \{ \sum \sigma_i(2n)Q_i(2n), \sum R_i(2n)Q_i(2n) \}$

Theorem. The algorithm is throughput optimal as long as probability of each $\sigma \in S$ is chosen with strictly positive prob. under random selection.
Proof. Lyapunov function:

\[ L(t) = Q(t) \cdot \dot{Q}(t) + \Delta^2(t), \]

\[ \Delta(t) = \left( \max_{\sigma \in S} \sigma \cdot Q(t) \right) - \left( \sigma(t) \cdot Q(t) \right) \]

\[ Q(t + h) = Q(t) - \sigma(t) \cdot \sum_{\tau} Q(\tau) \cdot \varphi_{\tau} + A(t + h). \]

\[ L(t + h) - L(t) = \left[ Q(t) + \dot{Q}(t) \right] \cdot \left[ Q(t) + \dot{Q}(t) \right] + \Delta^2(t + h) \]

\[ - Q(t) \cdot Q(t) - \Delta^2(t) \]

\[ = 2 \dot{Q}(t) \cdot \dot{Q}(t) + \dot{Q}(t) \cdot \dot{Q}(t) + (\Delta^2(t + h) - \Delta^2(t)) \]

Now, \( \delta(t) \cdot \dot{\delta}(t) \leq \eta \)

\[ \Delta(t + h) = \begin{cases} 0 \quad \text{w.p. } \geq \delta = \min_{\sigma \in S} P[R(t + h) > \sigma] \\ \leq \Delta(t) + \eta \quad \text{w.p. } 1 - \delta. \end{cases} \]
For $Q(\tau) \cdot \sigma(\tau)$:

$$\mathbb{E} \left[ Q(\tau) \cdot \sigma(\tau) \mid \mathcal{F}_\tau \right] = -Q(\tau) \cdot \left( \sigma(\tau) - \lambda \right)$$

$$= -\left( Q(\tau) \cdot \sigma(\tau) \right) + Q(\tau) \cdot \lambda$$

$$= \Delta(\tau) - \left\{ \max_\sigma \left( Q(\tau) \cdot \sigma \right) \right\} \left( 1 - \rho(\lambda) \right)$$

where $\lambda \leq \sum_{\sigma^*} \alpha_\sigma \cdot \sigma$; $\sum_{\sigma^*} \alpha_\sigma = \rho(\lambda) < 1$.

Therefore,

$$\mathbb{E} \left[ Q(\tau) \cdot \sigma(\tau) \mid \mathcal{F}_\tau \right] \leq -\frac{1 - \rho(\lambda)}{\eta} \left| Q(\tau) \right| + \Delta(\tau)$$

Putting above together:

$$\mathbb{E} \left[ L(\tau, \eta) - L(\tau) \mid \mathcal{F}_\tau \right] \leq -\frac{1 - \rho(\lambda)}{\eta} \left| Q(\tau) \right| + \Delta(\tau) - \delta \cdot \Delta^2(\tau)$$

$$+ 2\eta \Delta(\tau) + \eta^2 + \eta$$

--- *(x)*
Now, if $\Delta(z) > B \Rightarrow |\Omega(z)| > B$.

Therefore, the cost will not decrease in $\xi$ only if $\Delta(z)$ is small and $|\Omega(z)|$ is small.

$C_1 = \{ \Delta(z) \text{ small} \}$, $C_2 = \{ |\Omega(z)|, \text{ small} \}$.

$C = C_1 \cap C_2$. Then,

$\mathbb{E} [ L(z+1) - L(z) | F_z ] \leq -\epsilon + \frac{1}{u} \mathbb{1}_{\{ \text{state in } C \}}$

for $u > 0$ some constant, and $C$ a bounded set by definition.

Thus by Lyapunov–Foster’s criteria, it is throughput optimal. #.