1 Transformations [ /10]

1.1 Linearity [ /3]

What does it mean for a transformation or an operator to be linear? [ /3]

1.2 Homogeneous coordinates and IFS [ /7]

Consider the 2D IFS (Iterated Function System) defined in 2D by

\[ A = \bigcup f_i(A) \]

That is, this fractal is the set of points \( A \) that is equal to the union of its transformed versions by the transformations \( f_i \). The \( f_i \) are described by the following matrices in homogeneous coordinates

\[
\begin{align*}
    f_0 &= \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
    f_1 &= \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
    f_2 &= \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

Explain the effect of each of the three transformations (e.g. translation by something followed by a rotation by another thing). [ /4]
What is the resulting fractal? The name is not necessary, you can roughly draw it. Advice: draw the first few iterations starting with the unit square from $(0,0)$ to $(1,1)$.

2 Curves and surfaces

In class, we have focused on cubic Bézier splines. However, one can similarly define quadratic Bézier splines using the Bernstein polynomials:

\[
B_1(t) = (1-t)^2 \quad B_2(t) = 2t(1-t) \quad B_3(t) = t^2
\]

How many control points do we need for a quadratic Bézier spline? 

Prove that the weights defined by these basis functions always sum to one.

Why is it critical for splines that the weights sum to one?
Does the curve approximate or interpolate the control points? The answer can be different for the various points...

What is the derivative (tangent) at the two extremities as a function of the control point?

What does it mean geometrically? That is, how is the geometric tangent at the extremities related to the control points?

How many control points do we need for a tensor-product quadratic Bézier patch?
3 Animation

3.1 Particles

Consider a simplified 1D version of the spring equation: $\frac{d^2x}{dt^2} = -kx$ where $x$ is a scalar function. The initials conditions are $x(0) = d$ and $\frac{dx}{dt}(0) = 0$.

What is the rest length of this spring? [ /1]

Describe the system after one step of Euler integration with time step $h$ (that is, give the values of $x$ and $\frac{dx}{dt}(0) = 0$, which you can note $v$). [ /3]

Describe the system after two step of Euler integration with time step $h$. [ /3]

For which value of $h$ does the length of $x$ increase after two iterations compared to the initial length? That is, when do we have $|x_2| > |x_0|$, where $x_i$ is the value after $i$ iterations. [ /3]
3.2 Quaternion

Let $q_1$ and $q_2$ be two unit quaternions. Prove that $(q_1 q_2)^* = q_2^* q_1^*$.

First, prove this using quaternion algebra. Recall that $(d; \vec{u})^* = (d; -\vec{u})$ and $(d, \vec{u})(d', \vec{u}') = (dd' - \vec{u}.\vec{u}'; d\vec{u}' + d'\vec{u} + \vec{u} \times \vec{u}')$. [ /5]

Second, give a geometric or matrix argument. [ /3]

4 EXTRA CREDIT

4.1 Easy extra credit

Consider a general multivariable linear first-order ODE of the form $\frac{dX}{dt} = MX$ where $X$ is an $n$-dimensional vector and $M$ is an $n \times n$ matrix.

Derive the implicit Euler integration for this case. That is, express $X(t + h)$. 
4.2 Harder extra credit

What limits the stability of the method, that is, how does the maximum stable time step $h$ relate to properties of the matrix?

4.3 Even more fun extra credit

Now consider a general first-order multivariate ODE of the type $\frac{dX}{dt} = f(X)$ where $f$ is an arbitrary smooth function. How do you adapt the above implicit integration scheme to this situation?