• This is a closed book exam, but two 8½" × 11" sheets of notes (both sides) are allowed.

• Calculators are not allowed.

• Make sure you have all 16 numbered pages of this exam.

• There are 5 problems on the exam.

• The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best. Also note that very often the later parts of a problem can be done independently of earlier parts.

• A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and show all relevant work. Your grade on each problem will be based on our assessment of your level of understanding as reflected by what you have written in the space provided.

• Please be neat—we can’t grade what we can’t decipher.

• Only the booklet supplied is to be handed in. No additional pages will be considered in the grading. You may want to first work through the problems on the scratch paper provided and then neatly transfer the work you would like us to look at into the exam booklet. Let us know if you need additional scratch paper.

• We will be following a grading strategy that focuses on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess your level of understanding from what you have written.

Based on our judgement, each part will receive one of the following grades: E (EXCELLENT UNDERSTANDING), G (GOOD), R (REASONABLE), M (MARGINAL) or U (UNSATISFACTORY). Where appropriate, we may also use grades of E/G, G/R, R/M, or M/U so that there is a finer scale.

For each part of each exam question we indicate the allotted percentage of the total exam grade. Your numerical score on that part will then be calculated accordingly. (For example, if a part represents 16% of the exam grade, then E=16, E/G=14, G=12, G/R=10, R=8, R/M=6, M=4, M/U=2, U=0.)

This system (which we’ve used in the course many times in the past) is designed to focus your attention and ours on your level of understanding of the material, rather than on what calculations and miscalculations receive or lose points. Consequently, in looking at your graded exam, we ask that you focus on our judgement of your level of understanding. If you honestly feel that we’ve misjudged it based on what you’ve written in the exam booklet, then we’d be happy to discuss it with you.
<table>
<thead>
<tr>
<th>Problem</th>
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Problem 1  (15%)  

Each of the parts of this problem are independent. Be sure to justify your answers using clear, concise reasoning.

(5%) (a) A sequence $x[n]$ has a discrete-time Fourier transform $X(e^{j\omega})$. In terms of $X(e^{j\omega})$, determine $G(e^{j\omega})$, the discrete-time Fourier transform of 

$$g[n] = x[-2n + 1].$$

(5%) (b) $s[n]$, $x[n]$ and $w[n]$ are wide-sense stationary random processes where 

$$s[n] = x[n]w[n].$$

$x[n]$ and $w[n]$ are zero-mean and statistically independent. The autocorrelation function of $w[n]$ is 

$$E\{w[n]w[n + m]\} = \sigma_w^2 \delta[m],$$

and the variance of $x[n]$ is $\sigma_x^2$.

Show that $s[n]$ is white, with variance $\sigma_x^2 \sigma_w^2$. 
(5%) (c) Consider a length-$N$ sequence $x[n]$, i.e.

$$x[n] = 0 \text{ for } n > N - 1 \text{ and } n < 0.$$ 

The discrete-time Fourier transform of $x[n]$ is $X(e^{j\omega})$, and the $N$-point DFT of $x[n]$ is $X[k]$, i.e.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

and

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \text{ for } k = 0, 1, \ldots, N - 1.$$ 

If $\Re\{X[k]\} = 0$ for $k = 0, 1, \ldots, N - 1$, can we conclude that $\Re\{X(e^{j\omega})\} = 0$ for $-\pi \leq \omega \leq \pi$? If your answer is yes, explicitly show why. If not, give a simple counterexample.
Problem 2  \((22\%)\)

In Figure 2.1 is shown an impulse response \(h[n]\), specified as

\[
h[n] = \begin{cases} 
\left(\frac{1}{2}\right)^{n/4} u[n], & \text{for } n \text{ an integer multiple of 4} \\
\text{constant in between as indicated} & 
\end{cases}
\]

Figure 2.1

(5%) (a) Determine a choice for \(h_1[n]\) and \(h_2[n]\) such that

\[
h[n] = h_1[n] \ast h_2[n],
\]

where \(h_1[n]\) is an FIR filter and where \(h_2[n] = 0\) for \(n/4\) not an integer. Is \(h_2[n]\) an FIR or IIR filter?
The impulse response $h[n]$ is to be used in a downsampling system as indicated in Figure 2.2.

$$x[n] \rightarrow h[n] \rightarrow \downarrow 4 \rightarrow y[n]$$

Figure 2.2

(12%) (b) Draw a flow graph implementation of the system in Figure 2.2 that requires the minimum number of non-zero and non-unity coefficient multipliers. You may use unit delay elements, coefficient multipliers, adders and compressors. (Multiplication by a zero or a one does not require a multiplier.)

(5%) (c) For your system state how many multiplications per input sample and per output sample are required, with a brief explanation.
Problem 3  (20%)  

In our discussion of quantization following multiplication in the implementation of a digital filter, the quantization error was modeled as an additive zero-mean white noise source uncorrelated with the signal being quantized. This model often works well for fixed-point multiplication. With floating-point arithmetic the quantization noise scales with the signal being quantized. (See Section 6.8.5 in the text.)

Figure 3.1 depicts a floating-point quantizer and the corresponding model.

![Diagram of a floating-point quantizer](image)

Figure 3.1

$\varepsilon[n]$ is assumed to be wide-sense stationary, white, uniformly distributed between $\pm \Delta/2$, and statistically independent of $x[n]$, the input to the quantizer.

For the following questions, you can use the result stated in Problem 1(b).

(5%) (a) Determine the autocorrelation function $\phi_{\varepsilon\varepsilon}[m]$ of $\varepsilon[n]$, i.e. $\phi_{\varepsilon\varepsilon}[m] = E \{ \varepsilon[n+m] \varepsilon[n] \}$. 

$x[n]$ is assumed to be a zero-mean, wide-sense stationary random process with power spectral density $\Phi_{xx}(e^{j\omega})$ shown in Figure 3.2.

![Figure 3.2](image)

(4%) (b) Determine $E\{x^2[n]\}$.

(5%) (c) Determine the autocorrelation function of the quantization error $s[n] = (\hat{x}[n] - x[n])$. (Note that we have assumed that $\varepsilon[n]$ and $x[n]$ are statistically independent.)
(6%) (d) In the first-order system shown in Figure 3.3 below, a floating point quantizer is applied after the coefficient multiplication. An equivalent model to that of Figure 3.1 for the floating-point quantizer is as an additive, zero-mean, white noise source $s[n]$ that is statistically independent of $x[n]$ and $w[n]$ and has variance $\sigma_s^2 = \sigma_w^2 \sigma_e^2$. The coefficient $a$ has magnitude $|a| < 1$.

\[
\begin{align*}
x[n] & \quad \downarrow \quad + \quad \downarrow \quad a^{-1} \\
& \quad \downarrow \quad + \quad \downarrow \quad \sigma^{-1} \\
& \quad \downarrow \quad + \quad \downarrow \quad g_x[n] + g_q[n] \\
\end{align*}
\]

Figure 3.3

The input $x[n]$ is a zero-mean, wide-sense stationary random process with autocorrelation

\[
\phi_{xx}[m] = E\{x[n+m]x[n]\} = \delta[m].
\]

$g_x[n]$ denotes the output that would result if there were no quantization error, i.e. for $s[n] = 0$. Determine $E\{g_x^2[n]\}$. 
Problem 4  (23%) 

(7%) (a) Figure 4.1 is the ideal, desired frequency response amplitude for a bandpass filter to be designed as a Type I FIR filter $h[n]$, with DTFT $H(e^{j\omega})$ that approximates $H_d(e^{j\omega})$ and meets the following constraints:

\[-\delta_1 \leq H(e^{j\omega}) \leq \delta_1, \quad 0 \leq |\omega| \leq \omega_1\]

\[1 - \delta_2 \leq H(e^{j\omega}) \leq 1 + \delta_2, \quad \omega_2 \leq |\omega| \leq \omega_3\]

\[-\delta_3 \leq H(e^{j\omega}) \leq \delta_3, \quad \omega_4 \leq |\omega| \leq \pi\]

![Figure 4.1](image-url)

Figure 4.1

The resulting filter $h[n]$ is to minimize the maximum weighted error and therefore must satisfy the alternation theorem.

Determine and sketch an appropriate choice for the weighting function to use with the Parks-McClellan algorithm.
(8%) (b) Shown in Figure 4.2 is the frequency response $A(e^{j\omega})$ of a lowpass Type I Parks-McClellan filter based on the following specifications. Consequently it satisfies the alternation theorem.

<table>
<thead>
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<th>Specification</th>
<th>Value</th>
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<tr>
<td>Passband edge</td>
<td>$0.45\pi$</td>
</tr>
<tr>
<td>Stopband edge</td>
<td>$0.50\pi$</td>
</tr>
<tr>
<td>Desired passband magnitude</td>
<td>1</td>
</tr>
<tr>
<td>Desired stopband magnitude</td>
<td>0</td>
</tr>
</tbody>
</table>

The weighting function used in both the passband and the stopband is $W(\omega) = 1$.

What can you conclude about the maximum possible number of non-zero values in the impulse response of the filter?

(8%) (c) In Figure 4.3 is shown another frequency response $B(e^{j\omega})$ for a Type I FIR filter. $B(e^{j\omega})$ is obtained from $A(e^{j\omega})$ as follows:

$$B(e^{j\omega}) = k_1 (A(e^{j\omega}))^2 + k_2,$$

where $k_1$ and $k_2$ are constants. The filter specifications are the same as in Part (b). The weighting function on the error is constant in the passband and constant in the stopband but not the same in both bands.

Does this filter satisfy the alternation theorem with the passband and stopband edge frequencies indicated and with passband ripple and stopband ripple indicated by the dashed lines? As usual, provide clear and concise reasoning.
Figure 4.2: Frequency response $A(e^{j\omega})$. Dashed lines indicate maximum error.

Figure 4.3: Frequency response $B(e^{j\omega})$. Dashed lines indicate maximum error.
Problem 5  (20%)

In Figure 5.1 is shown an IIR lattice filter.

(5%) (a) Determine $y[1]$ for input $x[n] = \delta[n]$.

(7%) (b) Determine a flow graph for an inverse filter.
(8%) (c) Determine the transfer function for the IIR filter in Figure 5.1.
END OF EXAM
You may (or may not) find the following facts useful for this exam.

**Potentially useful concepts:**

1) Random Signals

   - $x[n]$ is wide-sense stationary (WSS) if
     - $\mathbb{E}\{x[n]\} = \mu$ independent of $n$ and
     - $\mathbb{E}\{x[n]x[n+m]\} = \phi_{xx}[m]$ independent of $n$.

   Power-spectral density (PSD) of WSS $x[n]$

   $$\Phi_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]e^{-jm\omega}$$

2) Fourier transform equation

   $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

   Inverse Fourier transform equation

   $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

3) Group delay of $H(e^{j\omega})$

   $$\tau[\omega] = \frac{d}{d\omega} \arctan[H(e^{j\omega})]$$

4) Continuous/discrete-time converters

   $x(t) \xrightarrow{C/D} x[n] \xrightarrow{T} x[n]$

   $$x[n] = x_d(nT)$$

   $$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right)$$

   $x[n] \xrightarrow{D/C} x(t)$

   $x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT)/T$

   $$X_c(j\Omega) = \begin{cases} \frac{T}{2\pi} X(e^{j\Omega T}), & -\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T}, \\ 0, & \text{otherwise}. \end{cases}$$

5) Expanders and compressors

   $x[n] \xrightarrow{\text{ Expander}} x[n] = x[nM]$

   $$X_d(e^{j\omega}) = \sum_{M=0}^{M-1} X(e^{j(M-2\pi/M)})$$
Expanders and compressors (con't)

\[ x[n] \xrightarrow{\gamma L} x_L[n] = \begin{cases} \frac{x[n/L]}{L} & \text{if } n/L \text{ is integer} \\ 0 & \text{otherwise} \end{cases} \]

\[ X_x(e^{j\omega}) = X(e^{j\omega L}) \]

6) Flow graph structures

- **Direct Form I**
- **Direct Form II**

7) Transposition theorem

1. Reverse direct of all branches
2. Interchange input and output
   Transfer function remains the same

8) FIR lattice

\[ A^{(l)}(z) = A^{(l-1)}(z) - k_l z^{-1} B^{(l-1)}(z) \]
\[ B^{(l)}(z) = -k_l A^{(l-1)}(z) + z^{-1} B^{(l-1)}(z) \]

\[ A^{(0)}(z) = z^{-1} B^{(0)}(1/z) \]
\[ B^{(0)}(z) = z^{-1} A^{(0)}(1/z) \]
9) Alternation Theorem

**Alternation Theorem**  
Let $F_p$ denote the closed subset consisting of the disjoint union of closed subsets of the real axis $x$. $P(x)$ denotes an $r$th-order polynomial

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

Also, $D_p(x)$ denotes a given desired function of $x$ that is continuous on $F_p$. $W_p(x)$ is a positive function, continuous on $F_p$, and $E_p(x)$ denotes the weighted error

$$E_p(x) = W_p(x) \left[ D_p(x) - P(x) \right]$$

The maximum error $|E|$ is defined as

$$|E| = \max_{x \in F_p} |E_p(x)|$$

A necessary and sufficient condition that $P(x)$ is the unique $r$th-order polynomial that minimizes $|E|$ is that $E_p(x)$ exhibit at least $(r+2)$ alternations, i.e., there must exist at least $(r+2)$ values $x_i$ in $F_p$ such that $x_1 < x_2 < \ldots < x_r < 2$ and such that $E_p(x_i) = -E_p(x_{i+1}) = \pm |E|$ for $i = 1, 2, \ldots, (r+1)$.
10) All-pole modeling

Inverse Approach

\[ s[n] = \frac{1}{A} \left[ 1 - \sum_{k=1}^{p} a_k z^{-k} \right] g[n] \]

Choose \( a_k \)'s so that \( g[n] \) approximates \( s[n] \)

\[ e[n] = g[n] - s[n] \]

Minimize \( \sum |e[n]|^2 \)

\[ \sum_{n=0}^{\infty} s[n][s[n-i] - \sum_{k=1}^{p} a_k s[n-k] - A \delta[-1] = 0 \]

Define \( \phi[n] = \sum_{n=0}^{\infty} s[n+m] s[n] \)

(deterministic autocorrelation of \( s[n] \))

\[ \sum_{i=1}^{p} a_i \phi[i-k] = \phi[i] - A \delta[-1] \quad \text{zero if } |i| \text{ cancel} \]

Autocorrelation Normal Equations

Choose gain factor \( A \) so that total energy of model equals total energy of \( s[n] \)

11) Noble identities

Equivalent compressing systems

\[ \begin{array}{c}
\downarrow M \\
x[n] \rightarrow x_d[n] \rightarrow H(z) \rightarrow y[n]
\end{array} \]

(a)

Equivalent expanding systems

\[ \begin{array}{c}
\downarrow L \\
x[n] \rightarrow \uparrow L
\end{array} \]

(b)