MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Electrical Engineering and Computer Science  
6.341 DISCRETE-TIME SIGNAL PROCESSING  
Fall 2008  

FINAL EXAM  

Wednesday, December 17, 2008  Johnson Track (W35) 1:30pm–4:30pm  

• This is a closed book exam, but **three** 8½” × 11” handwritten sheets of notes (both sides) are allowed.  
• Calculators are not allowed.  
• Make sure you have all 26 numbered pages of this exam.  
• There are 7 problems on the exam.  
• The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.  
• A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and **show all relevant work**.  
• Please be neat—we cannot grade what we cannot decipher.  
• Only this exam booklet is to be handed in. You may want to work things through on scratch paper first, and then neatly transfer the work you would like us to look at into the exam booklet. Let us know if you need additional scratch paper.  

WRITE YOUR ANSWERS ONLY IN THE DESIGNATED SPACES IN THIS EXAM BOOKLET. ANY WORK WRITTEN ANYWHERE ELSE, INCLUDING THE BLANK PAGES IN THIS BOOKLET, WILL NOT BE GRADED.  

• We will again be using the EGRMU grading strategy. This strategy focuses on your level of understanding of the material associated with each problem. Specifically, when we grade each part of a problem we will do our best to assess, from your work, your level of understanding.  

• **Graded Exams and Final Course Grade:**  
  Graded exams and final course grades can be picked up from Eric Strattman (in 36-615 or 36-680, depending on the time of day) on or after FRIDAY morning, December 19. If you would like your graded exam and project mailed to you, please leave an addressed, stamped envelope with us at the end of the exam. We will use the envelope as is, so please be sure to address it properly and with enough postage. We guarantee that we will put it into the proper mailbox, but we cannot guarantee anything beyond that.  

OUT OF CONSIDERATION FOR THE 6.341 STAFF, PLEASE DO NOT ASK FOR GRADES BY PHONE OR EMAIL.
NAME: _____________________________

<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
<th>Points</th>
<th>Grader</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>/5</td>
<td></td>
</tr>
<tr>
<td>2 (a)</td>
<td></td>
<td>/6</td>
<td></td>
</tr>
<tr>
<td>2 (b)</td>
<td></td>
<td>/6</td>
<td></td>
</tr>
<tr>
<td>3 (a)</td>
<td></td>
<td>/2</td>
<td></td>
</tr>
<tr>
<td>3 (b)</td>
<td></td>
<td>/10</td>
<td></td>
</tr>
<tr>
<td>3 (c)</td>
<td></td>
<td>/6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>/12</td>
<td></td>
</tr>
<tr>
<td>5 (a)</td>
<td></td>
<td>/3</td>
<td></td>
</tr>
<tr>
<td>5 (b)</td>
<td></td>
<td>/3</td>
<td></td>
</tr>
<tr>
<td>5 (c)</td>
<td></td>
<td>/5</td>
<td></td>
</tr>
<tr>
<td>5 (d)</td>
<td></td>
<td>/7</td>
<td></td>
</tr>
<tr>
<td>6 (a)</td>
<td></td>
<td>/6</td>
<td></td>
</tr>
<tr>
<td>6 (b)</td>
<td></td>
<td>/6</td>
<td></td>
</tr>
<tr>
<td>6 (c)</td>
<td></td>
<td>/5</td>
<td></td>
</tr>
<tr>
<td>7 (a)</td>
<td></td>
<td>/5</td>
<td></td>
</tr>
<tr>
<td>7 (b)</td>
<td></td>
<td>/8</td>
<td></td>
</tr>
<tr>
<td>7 (c)</td>
<td></td>
<td>/5</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>/100</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1  (5%) 

Determine and draw the lattice filter implementation of the following causal FIR system function:

\[ H(z) = 1 + \frac{1}{2}z^{-1} - \frac{3}{4}z^{-2} - \frac{1}{2}z^{-3} \]

Work to be looked at and answer:
THIS PAGE IS INTENTIONALLY LEFT BLANK. YOU CAN USE IT AS SCRATCH PAPER, BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING.
Problem 2  (12%) 

We find in a treasure chest an FIR bandpass filter \( h[n] \) that is zero phase, i.e. \( h[n] = h[-n] \). Its associated DTFT \( H(e^{j\omega}) \) is shown in Figure 2-1.

![Figure 2-1: Plot of \( H(e^{j\omega}) \) from \(-\pi \leq \omega \leq \pi\).](image)

The filter is known to have been designed using the Parks-McClellan (PM) algorithm. The input parameters to the PM algorithm are known to have been:

- Lower stopband edge \( \omega_1: 0.2\pi \)
- Lower passband edge \( \omega_2: 0.3\pi \)
- Upper passband edge \( \omega_3: 0.7\pi \)
- Upper stopband edge \( \omega_4: 0.8\pi \)
- Ideal passband gain \( G_p: 1 \)
- Ideal stopband gain \( G_s: 0 \)
- Error weighting function \( W(\omega) = 1 \)

The value of the input parameter \( N \), which represents the maximum number of non-zero impulse response values (equivalently the filter length), is not known.
Along with the plot is a fortune in gold, to be claimed by whoever can reproduce the filter with the Parks-McClellan algorithm for these specifications and with an appropriate value of \( N \). Multiple winners share the gold. You are the sole referee and judge.

Two entries have been submitted, each with a different associated value for the input parameter \( N \) to the algorithm.

- **Entry 1:** \( N = N_1 = 15 \)
- **Entry 2:** \( N = N_2 \neq N_1 \)

Both entrants claim to have obtained the required filter using exactly the same Parks-McClellan algorithm and input parameters, except for the value of \( N \).

After inspecting both entries, you determine that they both have DTFTs identical to Figure 2-1, so you deem both of them winners.

[6%] (a) What are possible values for \( N_2 \)?

**Work to be looked at and answer:**
(b) Both entrants claim that there can only be one winner, since the alternation theorem requires “uniqueness of the rth-order polynomial.” Explain why the alternation theorem is not violated.

Work to be looked at and answer:
THIS PAGE IS INTENTIONALLY LEFT BLANK. YOU CAN USE IT AS SCRATCH PAPER, BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING.
Problem 3  (18\%)

Consider the following system:

\[ x[n] \rightarrow h[n] \rightarrow \downarrow 4 \rightarrow y[n], \]

We want to implement this system using the following polyphase structure:

![Figure 3-1: Polyphase structure of the system.](image)

For parts (a) and (b) only, assume \( h[n] \) is defined as:

\[ h[n] = \begin{cases} 
0 & \text{for all } n < 0 \text{ and } n \geq 12, \\
\frac{1}{2} & \text{for } n = 0, 2, 4, 6, 8, 10, \\
\frac{1}{4} & \text{for } n = 1, 3, 5, 7, 9, 11.
\end{cases} \]
[2%] (a) Give the sequences $e_0[n]$, $e_1[n]$, $e_2[n]$, and $e_3[n]$ that result in a correct implementation.

Work to be looked at and answer:

[10%] (b) We want to minimize the total number of multiplies per output sample for the implementation of the structure in Figure 3-1. Using the appropriate choice of $e_0[n]$, $e_1[n]$, $e_2[n]$, and $e_3[n]$ from part (a), determine the minimum number of multiplies per output sample for the overall system. Also, determine the minimum number of multiplies per input sample for the overall system. Explain.

Work to be looked at and answer:
[6%] (c) Instead of using the sequences $e_0[n]$, $e_1[n]$, $e_2[n]$, and $e_3[n]$ you identified in part (a), now assume that $E_0(e^{j\omega})$, $E_1(e^{j\omega})$, $E_2(e^{j\omega})$, and $E_3(e^{j\omega})$, the DTFTs of $e_0[n]$, $e_1[n]$, $e_2[n]$, and $e_3[n]$, respectively, are as given below:

\[
E_0(e^{j\omega}) = \begin{cases} 1 & \text{for } -\omega_c < \omega < \omega_c \\ 0 & \text{else} \end{cases} \\
E_2(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r) \\
E_1(e^{j\omega}), E_3(e^{j\omega}) = 0
\]

Sketch and label $H(e^{j\omega})$ from $(-\pi, \pi)$.

Work to be looked at and answer:
Consider a 2000-point real discrete-time signal $x[n]$; i.e. $x[n] = 0$ for $n < 0$ and $n \geq 2000$. We want to obtain 4000 evenly spaced samples of the DTFT $X(e^{j\omega})$ of this signal; i.e. we want to obtain

$$X(e^{j\omega_k}) \bigg|_{\omega_k = \frac{2\pi k}{4000}}, \quad \text{for } k = 0, 1, \ldots, 3999. \quad (4.1)$$

The following method, depicted in Figure 4-1, is proposed to obtain a 4000-point sequence whose DFT gives the desired samples of $X(e^{j\omega})$.

**Proposed method:** First, $x[n]$ is delayed by 2000 samples to form $x_2[n]$. The delayed signal is then added to the original signal to form the sequence $x_p[n]$. The 4000-point DFT, $X_p[k]$, of this sequence then is computed.

(Continued on next page.)
Determine how $X_p[k]$ is related to $X(e^{j\omega})$. Indicate this relationship in a sketch for a “typical” Fourier transform $X(e^{j\omega})$. State explicitly whether this method provides the desired samples of $X(e^{j\omega})$ specified in (4.1).

Work to be looked at and answer:
Problem 5  (18%)

In the system shown in Figure 5-1, we want to process the input $x_c(t)$ with a first-order Butterworth lowpass filter; i.e.

$$H_c(s) = \frac{A}{s + a} = \frac{Y_c(s)}{X_c(s)}, \quad (5.1)$$

where $A$ and $a$ are constants.

This continuous-time LTI system $H_c(s)$ can be specified by the linear, constant-coefficient differential equation

$$\frac{dy_c(t)}{dt} + ay_c(t) = Ax_c(t). \quad (5.2)$$

Instead of implementing the filter directly in continuous time, we approximate $H_c(s)$ using the system shown in Figure 5-2.

Assume that the input $x_c(t)$ is bandlimited to $\pi/T$; i.e. $X(j\Omega) = 0$ for $|\Omega| \geq \pi/T$. We choose $H_d(z)$ in Figure 5-2 by approximating the derivative in (5.2) by the first-order central difference, defined as:

$$\left. \frac{dy_c(t)}{dt} \right|_{t=nT} \approx \frac{y_c(nT + T) - y_c(nT - T)}{2T}. \quad (5.3)$$

(Continued on next page.)
(3%) (a) Using the central difference approximation in (5.3), determine the difference equation relating $x_d[n]$ and $\hat{y}_d[n]$ in Figure 5-2, and find the associated system function $H_d(z)$.

Work to be looked at and answer:

(3%) (b) Specify $R(z)$, such that the mapping $s = R(z)$ results in $H_d(z) = H_c(s)\big|_{s=R(z)}$.

Work to be looked at and answer:
(5%) (c) For the mapping in part (b), determine the equation for the points in the $z$-plane that correspond to the points on the imaginary axis in the $s$-plane, given by $s = j\Omega$. For what values of $\Omega$, if any, do the corresponding points in the $z$-plane lie on the unit circle (i.e. $|z| = 1$)?

**Work to be looked at and answer:**
(7%) (d) Now, assume $H_c(j\Omega)$ is an ideal lowpass filter with cutoff frequency $\Omega_c = 10,000$ rad/s, with the frequency response shown in the figure below:

Using the relationship in part (c), draw the frequency response of the associated discrete-time filter $H_d(e^{j\omega})$ for the following sampling periods: (i) $T = 50 \mu s$ and (ii) $T = 0.2$ ms. For both cases, comment on the resulting frequency responses $H_d(e^{j\omega})$.

Work to be looked at and answer:
Problem 6  (17%)  

Figure 6-1 depicts a system for estimating $\phi_{xx}[m] = \mathcal{E}\{x[n]x[n + m]\}$, the autocorrelation of a Gaussian, zero-mean, WSS random process $x[n]$. The input signal $x[n]$ is first delayed by $m$ samples and multiplied by the (non-delayed) input, resulting in $b[n]$. The output signal $y[n]$, a sequence of estimates of $\phi_{xx}[m]$, is obtained by filtering $b[n]$ using a single-pole lowpass filter with a pole at $z = \alpha$.

For all parts, assume $0 \leq \alpha \leq 0.9$, and assume that $b[n]$ is WSS.

(a) Calculate the expected value of $b[n]$, i.e. $\mathcal{E}\{b[n]\}$. For what values of $n$ (if any) is $\mathcal{E}\{b[n]\} = \phi_{xx}[m]$, i.e. for what values of $n$ is $b[n]$ an unbiased estimate of $\phi_{xx}[m]$?

Work to be looked at and answer:
(6%) (b) Calculate $E\{y[n]\}$. For what value of $n$ and $\alpha$ (if any) is $E\{y[n]\} = \phi_{xx}[m]$, i.e. for what values of $n$ and $\alpha$ is $y[n]$ an unbiased estimate of $\phi_{xx}[m]$?

Work to be looked at and answer:
[5%] (c) What value for $\alpha$ ($0 \leq \alpha \leq 0.9$) will minimize the variance of $y[n]$? Explain.

Work to be looked at and answer:
THIS PAGE IS INTENTIONALLY LEFT BLANK. YOU CAN USE IT AS SCRATCH PAPER, BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING.
**Problem 7  (18%)**

In Figure 7-1, a filter bank is shown for which

\[ h_0[n] = 3\delta[n + 1] + 2\delta[n] + \delta[n - 1], \]

and

\[ h_q[n] = e^{j \frac{2\pi q n}{M}} h_0[n], \quad \text{for } q = 1, \ldots, N - 1. \]

The filter bank consists of \( N \) filters, modulated by a fraction \( M \) of the total frequency band. Assume \( M \) and \( N \) are both greater than the length of \( h_0[n] \).

---

**Figure 7-1: Filter Bank.**

---

[5%] (a) Express \( y_q[n] \) in terms of the TDDTFT \( X[n, \lambda] \) of \( x[n] \), and sketch and label explicitly the values for the associated window in the TDDTFT.

**Work to be looked at and answer:**
For parts (b) and (c), assume that $M = N$. Since $v_q[n]$ depends on the two integer variables $q$ and $n$, we alternatively write it as the two-dimensional sequence $v[q, n]$.

(b) For $R = 2$, describe a procedure to recover $x[n]$ for all values of $n$ if $v[q, n]$ is available for all integer values of $q$ and $n$.

Work to be looked at and answer:

5% (c) Will your procedure in (b) work if $R = 5$? Clearly explain.

Work to be looked at and answer:
END OF EXAM