NAME: ________________________________

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<th>Problem</th>
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Problem 1 [12%]

4% (a) Consider the system below, where the system is not trivially zero (there is at least one $\omega$ such that $H(e^{j\omega}) \neq 0$):

\[
x[n] \rightarrow \downarrow 2 \rightarrow H(e^{j\omega}) \rightarrow \uparrow 2 \rightarrow y[n]
\]

Explain which of the following is true:

A. This system is always LTI (LTI for any $H(e^{j\omega})$). In this case, give the system impulse response in terms of $h[n]$ or frequency response in terms of $H(e^{j\omega})$.

B. This system is sometimes LTI (LTI with some restrictions on $H(e^{j\omega})$). In this case, give the least-restrictive conditions for the system to be LTI.

C. This system is never LTI.
(b) Consider a signal $x[n] = s[n] \cos(0.3\pi n)$ where the envelope $s[n]$ is periodic with period $N$ and only has very low-frequency components. This signal is passed through a real lowpass filter $H_{lp}(e^{j\omega})$ with magnitude

$$|H_{lp}(e^{j\omega})| = \begin{cases} 1, & |\omega| < 0.5\pi; \\ 0, & \text{otherwise.} \end{cases}$$

In a narrow band around $\omega = 0.3\pi$, it is assumed the phase is approximately linear and can be written as

$$\arg[H_{lp}(e^{j\omega})] \approx -\phi_0 - \omega n_d.$$

If the output of the filter is denoted $y[n]$, which of the below conditions ensures $y[n] \approx x[n]$? Please provide succinct reasoning. Note that there may be more than one correct answer.

<table>
<thead>
<tr>
<th>$n_d$</th>
<th>$\phi_0$</th>
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<tbody>
<tr>
<td>A.</td>
<td>$N$</td>
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<tr>
<td>B.</td>
<td>$N$</td>
</tr>
<tr>
<td>C.</td>
<td>$2N$</td>
</tr>
<tr>
<td>D.</td>
<td>6</td>
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A stable and causal all-pole filter has transfer function

\[ H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{2}z^{-3}}. \]

Which could be the \( \gamma \) coefficients of the lattice implementation of this filter? Provide a short explanation.

A. \( \gamma = [1, \ -\frac{5}{7}, \ \frac{1}{4}, \ -\frac{1}{6}, \ \frac{1}{2}] \)
B. \( \gamma = [1, \ -\frac{4}{5}, \ -\frac{1}{6}, \ \frac{1}{2}] \)
C. \( \gamma = [1, \ -\frac{2}{3}, \ \frac{1}{5}, \ \frac{3}{4}] \)
D. \( \gamma = [1, \ -\frac{6}{5}, \ \frac{1}{16}, \ -\frac{1}{2}] \)

*Hint: You do not need to use the Levinson-Durbin recursion to solve this problem.*
Problem 2 [20%]
We have seen explicit and implicit periodic extensions of finite-length signals several times. Symmetric extensions are also quite common in signal processing. In the figure below, the top panel shows a periodic extension of a length-8 signal and the bottom panel shows a symmetric extension of the same signal.

3% (a) Let $x[n]$ be an $N$-point sequence and let $y[n]$ be the $2N$-point sequence obtained by symmetric extension of $x[n]$. Write $Y(z)$ in terms of $X(z)$. 
4% (b) Is symmetric extension a linear operation? Is symmetric extension a time-invariant operation?

3% (c) "Time reversal does not change a real wide-sense stationary random process." Support this statement by proving that if $x[n]$ is a real wide-sense stationary random process, $x[n]$ and $v[n] = x[-n]$ have the same autocorrelation sequence.
In light of part (c), it seems that the symmetric extension may be useful for spectrum estimation. The final two parts of this problem partially address this.

4% (d) Let $N$ samples $\{x[n]\}_{n=0}^{N-1}$ of some sequence $x[n]$ be given. Let $y[n]$ be the $2N$-point symmetric extension of $x[n]$. Prove that $Y(e^{j\omega})|_{\omega=\pi} = 0$ regardless of the spectrum of $x[n]$. 
Let $N$ samples $\{x[n]\}_{n=0}^{N-1}$ of a real wide-sense stationary process $x[n]$ be given. Let $y[n]$ be the $2N$-point symmetric extension of the given finite data segment. We consider here autocorrelation estimates computed from $\{y[n]\}_{n=0}^{2N-1}$. Specifically, define

$$
\hat{\phi}[m] = \frac{1}{2N-m} \sum_{n=0}^{2N-1-m} y[n+m]y[n], \quad \text{for } m = 0, 1, \ldots, 2N - 1.
$$

For which values of $m$, if any, is $\hat{\phi}[m]$ an unbiased estimate of $\phi_{xx}[m]$? For which values of $m$, if any, is $\hat{\phi}[m]$ an asymptotically unbiased estimate of $\phi_{xx}[m]$? Recall that $\hat{\phi}[m]$ is called an asymptotically unbiased estimate of $\phi_{xx}[m]$ when

$$
\lim_{N \to \infty} \mathbb{E} \left[ \hat{\phi}[m] \right] = \phi_{xx}[m].
$$
**Problem 3 [14%]**

Conventionally, a baseband signal $x(t)$ on a high-frequency carrier $e^{j\Omega_c t}$ is demodulated to baseband before sampling. The system below does *not* demodulate $x(t)e^{j\Omega_c t}$ before sampling and hence is called a *direct downconversion* or *direct downsampling* system:

![Diagram of Direct Downsampling System](image)

**Figure 3-1: Direct downsampling system**

The baseband signal $x(t)$ is a band-limited symmetric real signal with spectrum given in Figure 3-2. For this problem, assume $\Omega_c = 6\Omega_N$ and $T_s = \frac{4\pi}{\Omega_c}$.

![Spectrum Diagram](image)

**Figure 3-2: Spectrum of $x(t)$ and $y[n]$**

4% (a) Draw and label the continuous-time spectrum of $x(t)e^{j\Omega_c t}$.
4\% (b) Draw and label the discrete-time spectrum of $v[n]$.

6\% (c) In order to obtain $y[n]$ with spectrum as shown in Figure 3-2, what is the function of the black box? Draw a block diagram of the black box and label where appropriate.
**Problem 4 [8%]**
Consider a length-2048 signal that is known to be the sum of several sinusoidal components and a white noise process. We compute four spectral estimates using the following methods:

1. Periodogram
2. Bartlett’s method using a rectangular window and 16 segments
3. All-pole model of order 15
4. All-pole model of order 50

4% (a) The estimates are plotted below:

![Estimate A](image1)

![Estimate B](image2)

![Estimate C](image3)

![Estimate D](image4)

Match the techniques to the estimates and **briefly** justify your answer.
There is an unknown number of sinusoidal components (each with a different amplitude). Make a prediction on how many sinusoids there are. Justify your answer concisely using the four estimates given.

*Hint: Your reasoning is much more important than the actual prediction.*
Problem 5 [12%]

4% (a) The following two figures depict spectrograms of two short speech segments, where darker pixels represent areas of high energy. One segment was uttered by a typical adult female, and the other was uttered by a typical adult male. The parameters (length of time window, length of DFT, etc.) used to construct the spectrograms were the same.

![Spectrogram 1](image1)

![Spectrogram 2](image2)

Figure 5-1: Spectrogram 1

Figure 5-2: Spectrogram 2

Did the female utter the speech segment shown in Spectrogram 1 or 2? Explain your reasoning.
4% (b) The following figure depicts a spectrogram of a speech segment from one of the speakers from part (a):

![Spectrogram Image]

Is the speaker male or female? Explain your reasoning.

4% (c) The length of the time windows used to construct the spectrograms are different between parts (a) and (b). Which is longer? Briefly explain why it may be useful to construct spectrograms with different time window lengths.
Problem 6 [14%]
We wish to find the output $y[n]$ that results from passing an input signal $x[n]$ through a filter with impulse response $h[n]$. The input $x[n]$ is a length-$L$ sequence (as defined on the front matter). The filter $h[n]$ is a causal, $K$-tap FIR filter with $N$-point DFT $H[k]$. For all parts of the problem, $K \leq N$.

Our task is to determine $y[n]$ from the outputs of the following system:

In the diagram above, for any integer $\alpha$, the window $w_\alpha[n]$ is defined as

$$w_\alpha[n] = \begin{cases} 1, & 0 \leq n < \alpha; \\ 0, & \text{otherwise}. \end{cases}$$

Any processing of $v_0[n]$ and $v_1[n]$ to determine $y[n]$ cannot depend on knowledge of $h[n]$, but may depend on the system parameters $N, M, L$ and $K$.

4% (a) If $M = N$, what is the smallest choice of $N$ such that $y[n]$ may be determined from $v_0[n]$ and $v_1[n]$? Describe how to compute $y[n]$.
4% (b) Assume $N = L$. For what values of $n$ is $y[n] = v_0[n]$?

6% (c) Assume $N = L$ and $M < N$. Find lower and upper bounds on $M$ in terms of $N$ and $K$ such that $y[n]$ may be completely determined from $v_0[n]$ and $v_1[n]$. Describe how to compute $y[n]$. 
Problem 7 [10%]
The sequence $x[n]$ is a 16-point sequence, i.e., $x[n] = 0$ for $n < 0$ and for $n \geq 16$. Denote the DTFT of $x[n]$ by $X(e^{j\omega})$.

4% (a) Suppose that $X(e^{j(2\pi/16)k}) = 0$ for all integer values of $k$. Is $x[n] = 0$ for all $n$? If your answer is yes, provide compelling reasoning. If your answer is no, construct a counterexample.

6% (b) Now instead suppose that $X(e^{j(2\pi/16)(k+1/2)}) = 0$ for all integer values of $k$. Is $x[n] = 0$ for all $n$? If your answer is yes, provide compelling reasoning. If your answer is no, construct a counterexample.
**Problem 8 [10%]**
In this problem you will analyze and optimize the simplest possible critically-sampled modulated filter bank.

\[
x[n] \quad \begin{array}{c}
\downarrow\quad a + bz^{-1} \\
\end{array} \quad y_1[n] \\
\downarrow\quad a - bz^{-1} \\
\end{array}
\]

5% (a) Find a condition on \(a\) and \(b\) that ensures that the power of the output is equal to the power of the input for every \(\phi_{xx}[m]\). If no such condition exists, explain why.

Since the outputs \(y_1[n]\) and \(y_2[n]\) are at half the sample rate of the input \(x[n]\), by “equal power” we mean

\[
\sigma_x^2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2)
\]

where \(\sigma_x^2 = \mathbb{E}[(x[n])^2]\), \(\sigma_1^2 = \mathbb{E}[(y_1[n])^2]\), and \(\sigma_2^2 = \mathbb{E}[(y_2[n])^2]\).
The modulated filter bank is optimized for transform coding gain when the powers of $y_1[n]$ and $y_2[n]$ are as different as possible. Without loss of generality, make $\sigma_1^2 \geq \sigma_2^2$, so that the goal is maximize $\sigma_1^2 - \sigma_2^2$.

Let $x[n]$ be a wide-sense stationary random process with zero mean and autocorrelation

$$\phi_{xx}[m] = \frac{1}{4}\delta[m + 1] + \delta[m] + \frac{1}{4}\delta[m - 1].$$

Find coefficients $a$ and $b$ that optimize for transform coding gain while maintaining the total power as in part (a).