 FINAL EXAM INFORMATION  
Wednesday, December 16, 2009  
1:30pm–4:30pm  

• This is a closed-book exam, but three \(8\frac{1}{2}'' \times 11''\) sheets of (two-sided) handwritten notes are allowed.  
• Calculators are not allowed.  
• Make sure you have all numbered pages of this exam.  
• There are problems on the exam.  
• The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best. Often the later parts of a problem can be done independently of earlier parts.  
• In grading the exam, we will be doing our best to rate your level of understanding of the material. You best demonstrate your understanding by giving clear and concise answers. Every answer requires some justification, even if the question does not explicitly remind you to give a convincing argument.  
• Please be neat—we cannot grade what we cannot decipher.  
• You may want to work things through on the scratch paper provided first and then neatly transfer the work you would like us to look at into the exam booklet. Let us know if you need additional scratch paper.  
• We will be using the EGRMU grading system, which focuses on your level of understanding of the material associated with each problem. Specifically, when we grade each part of a problem we will do our best to assess your level of understanding on the scale of Excellent, Good, Reasonable, Marginal, or Unsatisfactory (with also the possibility of borderline assessments E/G, G/R, R/M, or M/U to provide a finer scale). This system (used in the course and several others many times in the past) is designed to focus your attention and ours on your level of understanding of the material, rather than on what calculations and miscalculations receive or lose points. Consequently, in looking at your graded exam, we ask that you focus on our judgment of your level of understanding. If you feel that we have misjudged your understanding—as evidenced by what you have written—we would be happy to discuss it with you. Rest assured that there is a logical (monotonic, consistent, \ldots) mapping from the EGRMU assessments to a final numerical score.  
• Graded Exams and Final Course Grade:  
Graded exams and final course grades can be picked up from Eric Strattman (in 36-615 or 36-680, depending on the time of day and day of the week) starting on Monday, January 4, 2010. If you would like your graded exam mailed to you, please leave a self-addressed, stamped envelope with us at the end of the exam. We will use the envelope as is, so please be sure to address it properly and with enough postage.
Definitions and Potentially-Useful Facts

Some trigonometric identities.

\[ \sin(2\theta) = 2\sin\theta\cos\theta \]
\[ \cos(2\theta) = \cos^2\theta - \sin^2\theta \]
\[ \cos^2\theta = \frac{1}{2} [1 + \cos(2\theta)] \]
\[ \sin^2\theta = \frac{1}{2} [1 - \cos(2\theta)] \]
\[ \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \]
\[ \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \]
\[ \tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \]
\[ \sin \alpha \sin \beta = \frac{1}{2} \left[ \cos(\alpha - \beta) - \cos(\alpha + \beta) \right] \]
\[ \cos \alpha \cos \beta = \frac{1}{2} \left[ \cos(\alpha - \beta) + \cos(\alpha + \beta) \right] \]
\[ \sin \alpha \cos \beta = \frac{1}{2} \left[ \sin(\alpha - \beta) + \sin(\alpha + \beta) \right] \]

**Wide-sense stationary.** A discrete-time random process \( x[n] \) is said to be wide-sense stationary when \( E[x[n]] \) and \( E[x[n+m]x^*[n]] \) have no dependence on \( n \). In this case, \( E[x[n+m]x^*[n]] \) is called the autocorrelation sequence and denoted \( \phi_{xx}[m] \). The DTFT of \( \phi_{xx}[m] \) is called the power spectrum and denoted \( \Phi_{xx}(e^{j\omega}) \).

**White random process.** A real, wide-sense stationary random sequence \( x[n] \) is called white if its autocorrelation is of the form \( \phi_{xx}[m] = \sigma^2_x \delta[m] \).

**Linear time-invariant (LTI).** A system is called linear and time-invariant (LTI) when

\[ x_1[n] \rightarrow \square \rightarrow y_1[n] \quad \text{and} \quad x_2[n] \rightarrow \square \rightarrow y_2[n] \]

implies

\[ Ax_1[n-a] + Bx_2[n-b] \rightarrow \square \rightarrow Ay_1[n-a] + By_2[n-b] \]

for any integers \( a \) and \( b \) and any numbers \( A \) and \( B \).

**Finite impulse response.** A filter \( h[n] \) is said to have finite impulse response when \( h[n] \) is nonzero for a finite number of values of \( n \).

**Causal.** A filter \( h[n] \) is said to be causal when \( h[n] \) is zero for all \( n < 0 \).

**Minimum phase.** An LTI system is said to be a minimum phase system if it is causal and stable and has a causal and stable inverse.
**All pass.** An LTI system with impulse response $h[n]$ and frequency response $H(e^{j\omega})$ is said to be an all-pass system when, for some constant $c$,

$$|H(e^{j\omega})| = c \quad \text{for all } \omega.$$ 

**BIBO stable.** A system is said to be stable in the bounded-input, bounded-output (BIBO) sense when every bounded input sequence produces a bounded output sequence.

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**Condition 1.** Suppose $H(e^{j\omega})$ is the frequency response of a discrete-time filter and can be expressed in polar coordinates as

$$H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$$

where $A(\omega)$ is even and real-valued and $\theta(\omega)$ is a continuous, odd function of $\omega$ for $-\pi < \omega < \pi$.

**Amplitude.** Under Condition 1, $A(\omega)$ is referred to as the amplitude.

**Unwrapped phase.** Under Condition 1, $\theta(\omega)$ is referred to as the unwrapped phase.

**Generalized linear phase.** Under Condition 1, if there are real constants $\alpha$ and $\beta$ such that

$$\theta(\omega) = \alpha + \beta \omega \quad \text{for } -\pi < \omega < \pi$$

then the filter is said to have generalized linear phase.

**Group delay.** Under Condition 1, the group delay $\tau(\omega)$ associated with the filter is defined as

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} \quad \text{for } |\omega| < \pi.$$ 

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**C/D and D/C conversion.** An ideal continuous-to-discrete time (C/D) converter operating with sampling period $T$ takes input $x_c(t)$ and produces output $x[n] = x_c(nT)$. An ideal discrete-to-continuous time (D/C) converter operating with sampling period $T$ can be understood as taking input $x[n]$, converting to an impulse train with spacing $T$, and filtering with an ideal lowpass filter with cutoff frequency $\pi/T$. (The C/D block does not include antialiasing filtering. The D/C block does include an ideal reconstruction filter.)

**Bilinear transformation.** The transformation

$$s = \frac{2}{T_d} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad \text{or} \quad z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}$$

is called the bilinear transformation.

**Type I FIR linear-phase systems.** A Type I FIR linear-phase system is one that has a real, symmetric impulse response

$$h[n] = \begin{cases} 
  h[M - n], & 0 \leq n \leq M; \\
  0, & \text{otherwise},
\end{cases}$$

with $M$ an even integer.
Type II FIR linear-phase systems. A Type II FIR linear-phase system is one that has a real, symmetric impulse response

\[ h[n] = \begin{cases} h[M - n], & 0 \leq n \leq M; \\ 0, & \text{otherwise}, \end{cases} \quad \text{with } M \text{ an odd integer.} \]

The z-transform of the impulse response \( H(z) \) must have a zero at \( z = -1 \).

Type III FIR linear-phase systems. A Type III FIR linear-phase system is one that has a real, antisymmetric impulse response

\[ h[n] = \begin{cases} -h[M - n], & 0 \leq n \leq M; \\ 0, & \text{otherwise}, \end{cases} \quad \text{with } M \text{ an even integer.} \]

The z-transform of the impulse response \( H(z) \) must have zeros at \( z = -1 \) and \( z = 1 \).

Type IV FIR linear-phase systems. A Type IV FIR linear-phase system is one that has a real, antisymmetric impulse response

\[ h[n] = \begin{cases} -h[M - n], & 0 \leq n \leq M; \\ 0, & \text{otherwise}, \end{cases} \quad \text{with } M \text{ an odd integer.} \]

The z-transform of the impulse response \( H(z) \) must have a zero at \( z = 1 \).

Alternation theorem. Let \( F_P \) be a disjoint union of closed intervals. Then \( P(x) = \sum_{k=0}^{r} a_k x^k \) is an \( r \)th-order polynomial. Also, \( D_P(x) \) denotes a given desired function of \( x \) that is continuous on \( F_P \); \( W_P(x) \) is a positive function, continuous on \( F_P \), and

\[ E_P(x) = W_P(x) [D_P(x) - P(x)] \]

is the weighted error. The maximum error is defined as

\[ \| E \| = \max_{x \in F_P} |E_P(x)|. \]

A necessary and sufficient condition that \( P(x) \) be the unique \( r \)th-order polynomial that minimizes \( \| E \| \) is that \( E_P(x) \) exhibit at least \( (r + 2) \) alternations; i.e., there must exist at least \( (r + 2) \) values \( x_i \) in \( F_P \) such that \( x_1 < x_2 < \cdots < x_{r+2} \) and such that \( E_P(x_i) = -E_P(x_{i+1}) = \pm \| E \| \) for \( i = 1, 2, \ldots, (r + 1) \).

Downsampling and upsampling identities.

\[ \begin{align*}
\downarrow M & \quad \rightarrow \quad H(z) \quad \rightarrow \quad \equiv \quad \rightarrow \quad H(z^M) \quad \rightarrow \quad \downarrow M \quad \rightarrow \\
\downarrow M & \quad \rightarrow \quad H(z) \quad \rightarrow \quad \uparrow L \quad \rightarrow \quad \equiv \quad \rightarrow \quad \uparrow L \quad \rightarrow \quad H(z^L) \quad \rightarrow
\end{align*} \]
Linear prediction. Let \( F_p(z) = \sum_{k=0}^{p-1} f_p[k] z^{-k} \) be designed to estimate wide-sense stationary random process \( s[n] \) from \( s[n - 1] \). When it minimizes the mean-squared error, it is called the optimal (one-step) (forward) predictor and \( A_p(z) = 1 - z^{-1} F_p(z) \) is called the optimal (forward) prediction error filter.

A resulting system of equations is the autocorrelation normal equations:

\[
\begin{bmatrix}
\phi_{ss}[0] & \phi_{ss}[1] & \cdots & \phi_{ss}[p - 1] \\
\phi_{ss}[1] & \phi_{ss}[0] & \cdots & \phi_{ss}[p - 2] \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{ss}[p - 1] & \phi_{ss}[p - 2] & \cdots & \phi_{ss}[0]
\end{bmatrix}
\begin{bmatrix}
f_p[0] \\
f_p[1] \\
\vdots \\
f_p[p - 1]
\end{bmatrix}
= \begin{bmatrix}
\phi_{ss}[1] \\
\phi_{ss}[2] \\
\vdots \\
\phi_{ss}[p]
\end{bmatrix},
\]

where \( \phi_{ss}[m] \) is the autocorrelation of \( s[n] \). The augmented normal equations are

\[
\begin{bmatrix}
\phi_{ss}[0] & r_p^T \\
r_p & \Phi_p
\end{bmatrix}
\begin{bmatrix}
1 \\
-f_p
\end{bmatrix}
= \begin{bmatrix}
P_p \\
0
\end{bmatrix},
\]

where \( P_p \) is the power of the error of the optimal \( p \)th order predictor.

An order-recursive algorithm for solving these equations is the Levinson–Durbin recursion. The key equations, using the notation from the lectures, are:

\[
\Delta_p = \sum_{k=0}^{p} a_p[k] \phi_{ss}[k - p - 1]
\]

\[
\gamma_{p+1} = -\Delta_p/P_p
\]

\[
a_{p+1} = \begin{bmatrix} a_p \\ 0 \end{bmatrix} + \gamma_{p+1} \begin{bmatrix} 0 \\ \tilde{a}_p \end{bmatrix} \quad \text{(the tilde represents time-reversal)}
\]

Discrete-time Fourier transform. The discrete-time Fourier transform (DTFT) of the sequence \( x[n] \) is defined by

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n},
\]

where the periodicity with respect to \( \omega \) and connection to the \( z \)-transform are suggested by the notation. When \( x[n] \) is such that the DTFT converges, the inversion formula

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega
\]

holds.

Discrete Fourier transform. The \( N \)-point discrete Fourier transform (DFT) of the sequence \( x[n] \) is defined by

\[
X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, 1, \ldots, N - 1,
\]

where \( W_N = e^{-j2\pi/N} \).
Inverse discrete Fourier transform. The $N$-point inverse discrete Fourier transform (IDFT) is defined by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad n = 0, 1, \ldots, N - 1,$$

where $W_N = e^{-j2\pi/N}$.

Time-dependent Fourier analysis. Given a window sequence $w[m]$, the time-dependent Fourier transform of $x[n]$ is defined as

$$X[n, \lambda] = \sum_{m=-\infty}^{\infty} x[n + m] w[m] e^{j\lambda m}.$$

Assume the support of the window sequence is $\{0, 1, \ldots, L - 1\}$. The time-dependent discrete Fourier transform is defined as

$$X[n, k] = X[n, \frac{2\pi k}{N}] = \sum_{m=0}^{L-1} x[n + m] w[m] e^{j(2\pi k/N)m}.$$

Subsampling the window shifts by $R$ gives the sampled time-dependent discrete Fourier transform

$$X_r[k] = X[rR, k] = X[rR, \frac{2\pi k}{N}] = \sum_{m=0}^{L-1} x[rR + m] w[m] e^{j(2\pi k/N)m}.$$

A plot of the magnitude or magnitude squared of the time-dependent Fourier transform or its sampled versions is called a spectrogram.

$N$-point sequence. A sequence $x[n]$ is said to be an $N$-point sequence if $x[n] = 0$ for all $n < 0$ and for all $n > N - 1$.

Periodogram, etc. The periodogram of the $N$-point sequence $x[n]$ is defined as

$$\frac{1}{N} |X(e^{j\omega})|^2,$$

where $X(e^{j\omega})$ is the discrete-time Fourier transform of $x[n]$. When in addition window $w[n]$ is supported on $\{0, 1, \ldots, N - 1\}$ and $\sum_{n=0}^{N-1} (w[n])^2 = 1$, then the periodogram of $x[n] w[n]$ is called the modified periodogram of $x[n]$. When $N$ is an integer multiple of integer $K$, the average of the $K$ periodograms of $(N/K)$-point nonoverlapping segments of $x[n]$ is called the averaged periodogram or Bartlett’s method. Similar averaging with overlapping segments is called Welch’s method.

Entropy rate. The (differential) entropy rate of a wide-sense stationary Gaussian random process $x[n]$ with power spectrum $\Phi_{xx}(e^{j\omega})$ is

$$H(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \Phi_{xx}(e^{j\omega}) \, d\omega.$$

Maximum entropy method. Given some values of the autocorrelation $\phi_{xx}[m]$ of wide-sense stationary process $x[n]$, the maximum entropy method for power spectrum estimation is to extrapolate $\phi_{xx}[m]$ to maximize the entropy. When the given values of $\phi_{xx}[m]$ are for $|m| \leq p$, the result is an all-pole spectrum of order $p$. 